

# Optimal Fragile Financial Networks\*

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## Abstract

In this paper we study the endogenous formation of a financial (banking) network. Establishing connections in the financial network is beneficial since banks, which invest on behalf of the depositors, can coinsure their uncertain liquidity needs. However, the network is fragile since banks have an incentive to gamble with depositors' money when not sufficiently capitalized. The bankruptcy of a bank negatively affects the banks connected to it in the network (counterparty risk). It turns out that both the efficient financial network and the decentralized one are characterized by a core-periphery structure. Nevertheless, the two network structures can have different degrees of connectivity. Under the assumption that bank capital transfers are not possible, we show that the two structures coincide if the counterparty risk is sufficiently low. Otherwise, the decentralized network is under-connected as compared to the optimal one. Finally, we show that bank capital transfers do not avoid the formation of inefficient financial networks.

**JEL Classification:** D85, G21.

**Keywords:** Financial Network, Core-periphery structure, Liquidity Coinsurance, Counterparty Risk, Financial Fragility.

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# 1 Introduction

Sometimes financial systems turn out to be fragile. This means that an adverse shock can cause the collapse of the entire financial system, implying large losses for investors and consumers. As a consequence, it is usually claimed that fragility should be avoided altogether. This paper instead establishes conditions under which such fragility is indeed an optimal feature of financial networks. Once the benchmark is obtained, we show under what circumstances agents have incentives to form an inefficient decentralized network.

The financial network is composed of several banks. Each bank is made of two types of agents: depositors and shareholders. While the former deposit their endowment into banks to take advantage of the investment opportunities in the economy, the latter provide bank capital and decide the type of investment the bank chooses. Banks have two types of projects in which they can invest. One project is safe and the other one is risky. The second project is risky since it has the same expected payoff as the safe project if it succeeds but it delivers nothing if it fails. However, it gives private benefits to the bank's shareholders. The second project can be considered a gambling project for the depositors. Given that shareholders are protected by limited liability, poorly capitalized banks find it convenient to invest in a gambling project.

Banks also have the possibility to join the financial network, whose structure is fully anticipated. Participating in the network is beneficial for banks since it allows them to coinsure future uncertain liquidity needs. Indeed, the investment projects (independently of their type) are affected by liquidity shocks before they mature. Then the links in the financial network represent contingent credit lines that banks grant to each other. Clearly, the decision of belonging to the network entails a trade-off: The benefits coming from the liquidity coinsurance have to be weighted against the possible bankruptcy of a bank that gambles (i.e., the counterparty risk). Indeed, when a bank fails all the directly linked banks have a lower probability to serve their depositors (that is, a higher probability of being bankrupt as well).

First we characterize the optimal financial network as the solution of the planner problem. Given the level of bank capital, the social planner can guarantee the unconstrained first-best if a sufficient amount of bank capital is available in the economy. In this way, the planner avoids the moral hazard problem in all banks, and the efficient network is the complete one since only the benefits of liquidity coinsurance matter. Otherwise, the planner allows some banks to gamble and a constrained first-best (CFB) is achieved.

The CFB network is characterized by a core-periphery structure. Indeed, since the bankruptcy transmission depends on the links established by the failing bank, the CFB

network structure is not necessarily fully connected. The core includes all the banks that invest in the safe project and form a complete network structure among themselves. The periphery includes all the gambling banks that can be connected among themselves and/or with the core banks according to the parameter's values. For given aggregate bank capital, if the counterparty risk is sufficiently low the CFB structure is the fully connected one. Otherwise, the higher the counterparty risk and the less connected the periphery banks are.

Second we analyze the decentralized network formation game in which banks decide whether to establish links among themselves, determining in this way the shape of the network. Also in this case a core-periphery structure emerges as an equilibrium outcome. Nevertheless, the connectivity in the decentralized network does not necessarily coincide with the CFB network. We show that the structure of the decentralized financial network is the same as the CFB one if the counterparty risk is low. Even if the decentralized network has the same structure as the efficient one, it could be that some banks invest in the gambling project in the decentralized network while they choose the safe project in the efficient one. That is, the core of the efficient network can be larger than the core of the decentralized network. However, when the counterparty risk is arbitrarily low, the structure of the network matters more than the type of investments made in determining the expected payoffs in the economy. Accordingly, the decentralized network delivers an expected payoff arbitrarily close to the efficient one.

On the other hand, when the counterparty risk is not low enough, the decentralized network does not coincide with the CFB network. The reason is that the social planner finds it optimal to link a safe bank with a gambling bank when the expected losses of the former (because of counterparty risk) are lower than the expected gains of the latter (due to liquidity coinsurance). However, these expected gains are not internalized by the safe banks that sever the link with the gambling banks in the decentralized network, even when this is not efficient. That is, the decentralized network is characterized by an inefficiently low degree of connectivity compared to the CFB network.

Finally, we allow for decentralized bank capital transfers in order to solve the moral hazard problem. That is, a bank investing in the safe project could find it convenient to transfer part of its bank capital to a neighboring gambling bank in 'exchange' for financial stability (and to achieve an higher expected payoff). For example, we show that allowing for such bank capital transfers, safe banks do not internalize the network externality identified above. This implies that the decentralized network formation does not induce banks to form the CFB network, even allowing for bank capital transfers.

The main empirical prediction of our model is that financial networks should have a

core-periphery structure. This is strongly sustained by recent evidence about the shape of banking networks. Based on transaction data from the Fedwire system, Soromäki et al. [24] and Beck and Atalay [6] find that the actual interbank lending networks formed by US commercial banks is quite sparse. It consists of a core of highly connected banks, while the remaining peripheral banks connect to the core banks. An almost identical feature is found in banking networks in other countries like the UK, Canada, Japan, and Austria (see, respectively, Bank of England [5], Embree and Roberts [12], Inaoka et al. [18], Boss et al. [8]). To the best of our knowledge this is the first paper that attempts to rationalize this evidence.

In this paper we analyze the contagious effect of the links *directly* established among banks. Accordingly, we focus on the externalities that arise from the linkages that banks voluntarily establish. We rule out from the analysis indirect contagion and systemic effects.<sup>1</sup> However, even with our narrow definition of contagion, the paper highlights that the beneficial role played by the financial network in providing liquidity insurance to its participants is negatively affected when counterparty risk is high.

This ineffectiveness has been evident during the 2007/2008 financial crisis, where banks feared losses in their counterparts. Banks became reluctant to lend to one another, and soon the market for short term lending dried up. This paper may help to explain this kind of phenomena. When the risk associated with the lending of funds is too high, connections become too costly relative to the benefits they bring and safe banks inefficiently sever their financial links. Nevertheless, the paper also stresses the fact that the liquidity insurance mechanism does not disappear altogether. Safe banks still continue to be linked and to insure one another. This prediction is supported by Afonso et al. [1] who provide evidence on interbank lending in the US during the 2007/2008 crisis. They find that interbank lending was very much stressed but it did not freeze completely. As our model predicts, they find that riskier banks were cut off from the interbank market while safer banks still would find coinsurance.<sup>2</sup> Moreover, they show that the interbank market stress was likely coupled with inefficient provision of liquidity to the risky banks.

Our paper is closely related to the strand of literature that model contagion as the outcome of links established by banks. In particular, banks are connected through interbank deposit markets that are desirable ex-ante, but the failure of one institution can have negative payoff effects on the institutions to which it is linked (see Allen and Gale, [3];

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<sup>1</sup>For example, the bankruptcy of a certain number of banks can cause the collapse of the entire financial network because system-wide network externalities could affect the payment system (see Freixas, Parigi and Rochet [13] and Kahn and Santos [20]).

<sup>2</sup>Notice that theoretical explanations of the market freeze based on adverse selection arguments predict that only the risky banks (i.e., the ‘lemons’) remain in the market.

Freixas, Parigi and Rochet, [13]; Brusco and Castiglionesi [9]). The common feature of all these models is to assume an exogenous banking network. In Allen and Gale [3] financial contagion is due to an unexpected aggregate liquidity shortage, accordingly the more connected the banking network is and the more resilient the system is to contagion. Freixas, Parigi and Rochet [13] model financial contagion as an unexpected solvency shock to a particular bank, and they also conclude that the degree of interbank connections enhance the resiliency of the banking system.

Brusco and Castiglionesi [9] instead model contagion in the banking system without relying on unexpected shocks. In their model, the possibility of a bank's bankruptcy comes from the banks' incentive to gamble when they are not sufficiently capitalized. They find that the more connected the banking system is, the larger the extent of contagion. The present model captures the features of the banking models (such as the benefits stemming from liquidity coinsurance as in Allen and Gale [3], and the gambling behavior of low capitalized banks as in Brusco and Castiglionesi [9]), but it directly addresses the issue of the endogenous network formation.

Indeed, even if the theory of network formation has been successfully applied to several economics fields, few attempts have been made to use such theory to understand the working of financial systems (see Allen and Babus [2] for a recent survey). Among those attempts, Leitner's [22] and Babus [4] consider models of endogenous network formation, where banks form links in order to reduce the risk of contagion. These models formalize the financial network as an insurance mechanism. The idea is that banks can be surprised by unexpected liquidity shocks that can bankrupt at least one bank in the system (like in Allen and Gale, [3]). The possibility that this original failure can spread to the entire system gives the rationale of belonging to a financial system. We provide an alternative rationale for financial network formation that is based on banks that fully anticipate the trade-off between the benefits and the costs of participating in the financial network.

Nier et al. [23] consider the link between network models and financial stability. Their approach is to take the network structure as given and study how an exogenous shock is transmitted through the network. Gai and Kapadia [16] study an analytical model of contagion in financial networks with an arbitrary structure assuming that the network forms randomly, leaving aside issues related to the endogenous network formation and its optimal structure. Gale and Kariv [17] show that trade that is restricted to happening on a network generates an efficient outcome as long as the agents are sufficiently connected in the network.

The paper is organized as follows. The section 2 sets up the model. The section 3 analyzes the planner problem, characterizing the constrained first-best solution. The

section 4 studies the decentralized network formation and its efficiency properties. The section 5 contains the conclusions. Appendix A provides an analysis of the microfoundation of the model and Appendix B contains the proofs.

## 2 The model

There are five dates  $t = 0, 1, 2, 3, 4$  and one divisible good called ‘dollars’ (\$). The economy is divided into  $n$  regions, each with its own representative bank. Let  $N = \{1, 2, \dots, n\}$  be the set of the regions and banks. Each region is populated by a continuum of consumers endowed with 1 \$ at  $t = 0$ . However, they consume at  $t = 4$ . In order to access the investment opportunities of the economy, each consumer has to deposit his or her endowment in the representative bank of the region that he or she belongs to.

Each representative bank  $i$  randomly receives an endowment  $e_i \in [0, \bar{e}]$  of dollars, which represents the bank capital and it is owned by the bank’s shareholders (or investors). Consumers and investors are different types of agents in the economy. The pair  $(N, e)$ , with  $N = \{1, 2, \dots, n\}$  and  $e = (e_1, e_2, \dots, e_n)$  is called an *economy*. Let  $K_i \subseteq N$  be the set of banks to whom bank  $i$  is directly linked, then the number of banks connected to bank  $i$  is  $k_i \in \{0, 1, \dots, n - 1\}$ . The vector  $K = (K_1, K_2, \dots, K_n)$  captures the interdependence among the banks, and it represents the financial network. We restrict ourselves to undirected networks, i.e., bank  $i$  is related to bank  $j$  if and only if bank  $j$  is related to bank  $i$ . Formally,  $i \in K_j$  if and only if  $j \in K_i$ . Let  $\mathcal{K}$  denote the set of all possible financial networks for a given economy  $(N, e)$ .

We also allow for transfer of bank capital across neighboring banks. The rationale of this transfers is to give banks an instrument that can induce other banks to reduce the incentive to gamble. The cost of transferring bank capital is compensated by having safer neighboring banks which implies an higher expect payoff. Let  $x_i = e_i + t_i$  be the bank capital for bank  $i \in N$  after transfers have been made (i.e.,  $t_i$  is the transfer and can be positive or negative). Then each bank expects to have  $1 + x_i$  dollars to invest. A vector of bank capitals  $x = (x_1, x_2, \dots, x_n)$  is called feasible for a given economy  $(N, e)$  if (i)  $x_i \geq 0$  for all  $i$ , and (ii)  $\sum_{i \in N} x_i = \sum_{i \in N} e_i$ . Let  $\mathcal{X}$  denote the set of all feasible vectors of bank capital for a given economy  $(N, e)$ . The sequence of events is reported in Table 1.

**Table 1. Sequence of events**

Time	Events
$t = 0$	Bank's capital is realized and financial network is chosen.
$t = 1$	Bank's capital transfers are made and projects are chosen.
$t = 2$	Banks offer deposit contracts to depositors.
$t = 3$	Idiosyncratic liquidity shocks hit the projects.
$t = 4$	Projects cash flows are realized and depositors are paid.

The timing from  $t = 0$  to  $t = 2$  represents the network formation game and it is the focus of analysis in this paper. In  $t = 0$ , once bank capitals are realized, the banks choose the network structure. In  $t = 1$  bank capital transfers are made and the investors choose the type of project. In  $t = 2$  banks offer to their depositors the deposit contract with a certain expected payoff, which is affected by the occurrence of idiosyncratic liquidity shocks. That is, the projects need to be refinanced in  $t = 3$  before they arrive at maturity. If they are not refinanced the investment is lost. Accordingly, the expected payoff in  $t = 2$  will depend on how much coinsurance banks have established in  $t = 0$  (i.e., on the financial network chosen) and on the type of project chosen by the banks in  $t = 1$ .

Banks can invest either in a liquid storage technology to face the liquidity shocks in  $t = 3$ , or in illiquid projects that mature in  $t = 4$ . Banks have access to two types of illiquid projects (the investment can be made either in one or the other type of project):

1. The safe project  $b$  has an expected return of  $\bar{R} > 1$  dollars in  $t = 4$  per dollar invested in  $t = 1$ .
2. The ‘gambling’ project  $g$  yields an expected return of  $\bar{R} > 1$  dollars with probability  $\xi$ , and 0 dollars with probability  $(1 - \xi)$ , in  $t = 4$  per dollar invested in  $t = 1$ . This type of project yields also a private benefit  $B > 0$  to bank's shareholders. Private benefits are realized by banks' shareholders at the moment of the investment (so they do not have dollar value, consider them as perks or investment in family business).

The timing captures the fact that bank decisions about which financial link to establish are long term, while the deposit contract offered to depositors is a short term decision (possibly repeated in the future).<sup>3</sup> On the one side, the choice of the financial network allows the banks to smooth future idiosyncratic liquidity shocks affecting the investment projects. On the other side, it exposes the banks to the possible gambling behavior of their neighbors. We analyze these two aspects in turn.

<sup>3</sup>There is strong evidence that documents how relationship lending plays a crucial role in providing liquidity to banks (see, for example, Fufine [14], King [21] and Cocco et al. [11]), a role which is also retained in times of crisis (see Fufine [15]).

## 2.1 Network Structure and Liquidity Coinsurance

Banks are exposed to independent projects' idiosyncratic liquidity shocks in  $t = 3$  so they find it convenient to establish links in the financial network. We can think of those links as credit lines that banks grant to each other. Assume that each bank in  $t = 3$  can face two equally likely liquidity shocks in their projects, either  $\omega_H$  or  $\omega_L$  with  $1 > \omega_H > \omega_L > 0$ . If the bank does not have enough liquidity to face the shock, the return of the project is lost. Consider a generic bank  $i$  in autarky, that is, without any links with other banks. In this case the bank has no possibility to coinsure against the idiosyncratic shocks. Accordingly, if we indicate with  $\varphi_i$  the probability of bank  $i$  getting coinsurance, in autarky we have  $\varphi_i = 0$ . Call  $R$  the expected return on the bank portfolio in autarky at  $t = 4$ . Clearly we have  $R < \bar{R}$  since banks, for each dollar collected, have to invest part of it in liquidity so the return  $\bar{R}$  cannot be reached.

Assume now that bank  $i$  has the possibility to coinsure with a second bank. Now we have four states of nature: in two states the banks have the same shock, but in the other two states they have negatively correlated liquidity shocks so coinsurance is possible. Since states are equally likely  $\varphi_i$  increases from 0 to  $1/2$ . Given that bank  $i$  has a higher probability of coinsurance, its expected payoff can be higher than the autarky return  $R$ . Assume now that bank  $i$  establishes a link with a third bank. In this case we have eight states of nature: only when the three banks have the same shock, bank  $i$  cannot find coinsurance. Otherwise, bank  $i$  can coinsure with one of the two linked banks. This increases the probability to coinsure up to  $\varphi_i = 3/4$ , and it can further increase the expected autarky return  $R$ .<sup>4</sup>

The reasoning can be generalized for  $k_i$  neighboring banks. In this case, the states of nature are  $2^{k_i+1}$ . Then

$$\varphi_i(k_i) = \frac{2^{k_i+1} - 2}{2^{k_i+1}} = 1 - 2^{-k_i},$$

and we have that  $\varphi_i'(k_i) > 0$  and  $\varphi_i''(k_i) < 0$ . Clearly, the probability of finding a neighbor with a different liquidity shock is increasing in the number of links. However, the higher the number of connections already in place the less valuable the marginal neighbor in terms of coinsurance. The function  $\varphi_i(k_i)$  represents the benefits from liquidity coinsurance that a financial network offers to banks and their depositors.

We assume a competitive banking system, then depositors take the full gains from liquidity coinsurance. Then depositors' expected payoff in bank  $i$  is

$$D_i = [1 + \varphi_i(k_i)]R, \tag{1}$$

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<sup>4</sup>In Appendix A we show what is the optimal investment decision in liquidity when there is the possibility of coinsurance.

where  $D_i$  is the expected amount of money delivered by the deposit contract.<sup>5</sup> In autarky ( $k_i = 0$ ) the expected amount is clearly  $R$ , otherwise the higher the number is of connections of bank  $i$  and the higher the probability is of finding coinsurance then the higher the expected amount of money is. We define

$$f(k_i) \equiv 1 + \varphi_i(k_i),$$

then it follows that  $f'(k_i) > 0$  and  $f''(k_i) \leq 0$  for all  $k_i \in [0, n - 1]$ . We assume that  $f(n - 1) = \rho < 2$ , that is  $f(k_i) \in [1, \rho]$ .

## 2.2 Network Structure and Counterparty Risk

Let  $s_i \in \{b, g\}$  be the project's choice of bank  $i$  at  $t = 1$ . The vector  $s$  denotes the investment strategy profile, that is,  $s = \{s_i\}_{i \in N}$ . Let  $\mathcal{S}$  denote the set of all possible investment profiles for a given economy  $(N, e)$ . For given network  $K \in \mathcal{K}$  and strategy profile  $s \in \mathcal{S}$ , let  $p_i(K, s)$  be the probability that bank  $i$  does not fail (i.e., it is able to serve its depositors the amount  $D_i$ ). This probability needs to take into account the possible use of the gambling asset in the financial network  $K$ . We assume

$$p_i(K, s) = \begin{cases} \prod_{j \in K_i} \pi_j(s_j) & \text{if } s_i = b, \\ \xi \prod_{j \in K_i} \pi_j(s_j) & \text{otherwise,} \end{cases} \quad (2)$$

where  $\pi_j(s_j)$  is defined as

$$\pi_j(s_j) = \begin{cases} 1 & \text{if } s_j = b, \\ \eta & \text{otherwise.} \end{cases}$$

Note that, when bank  $i$  and all its neighbors are investing in the safe project  $b$ , then bank  $i$  serves its depositors with probability 1. However, if one of bank  $i$ 's neighbors is investing in the gambling project, bank  $i$  will serve its depositors with probability  $\eta$ . If bank  $i$  has two neighbors who are investing in the gambling project, then the probability of serving its depositors reduces to  $\eta^2$ , and so on. In Appendix A we show  $\eta > \xi$ , which means that a gambling bank fails with a probability  $(1 - \xi)$  higher than the probability  $(1 - \eta)$  of the failure of its neighbors. The parameter  $\eta$  then captures the counterparty risk of establishing a link with a risky bank.

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<sup>5</sup>In Appendix A, we show under what conditions the expected payoff of the deposit contract  $D_i$  is increasing in the probability of finding coinsurance  $\varphi_i(k_i)$ .

## 2.3 Expected Payoffs and Investment Project Choice

Given the possible presence of gambling banks in the network, depositors in bank  $i$  have the following expected payoff

$$M_i(K, s) = p_i(K, s)D_i = p_i(K, s)f(k_i)R$$

for their dollar deposited in the bank. We assume that  $\xi R \geq 1$ , so that depositors will participate in a gambling bank in autarky. Notice that among the strategies of the banks there is no possibility of avoiding the investment. Since the banks can choose to be connected or not to the network, it is always possible for them to be disconnected and invest in autarky.

Investors are residual claimant. Accordingly, investors in bank  $i$  choosing a strategy  $s_i$  expect the following payoff:

$$m_i(K, x_i, s) = \begin{cases} p_i(K, s)[(1 + x_i)f(k_i)R - D_i], & \text{if } s_i = b, \\ p_i(K, s)[(1 + x_i)f(k_i)R - D_i] + B, & \text{otherwise.} \end{cases}$$

The investors' expected payoff in bank  $i$  is determined by the bank capital after transfers (that is,  $x_i = e_i + t_i$ ), the financial network  $K$  chosen at date 0, the strategies of all the other banks  $s_{-i}$ , with  $s = (s_i, s_{-i})$ , and the amount of money promised in the deposit contract. Furthermore when investors in bank  $i$  gamble, they obtain private benefits  $B$  independently of the gambling asset being successful or not. Notice that investors are protected by limited liability: When the bank is bankrupt, which happens with probability  $1 - p(K, s)$ , investors do not have to pay anything more than their bank capital. The investors' expected payoff, considering the amount  $D_i$  to be paid to depositors, can be written as follows

$$m_i(K, x_i, s) = \begin{cases} p_i(K, s)f(k_i)Rx_i, & \text{if } s_i = b, \\ p_i(K, s)f(k_i)Rx_i + B, & \text{otherwise.} \end{cases} \quad (3)$$

For the given network  $K$  and strategy profile  $s = (s_i, s_{-i})$ , let  $g_i(K, s_{-i})$  denote the number of gambling neighbors of bank  $i$ . To avoid abuse of notation, we will make use of  $g_i$  instead of  $g_i(K, s_{-i})$ , unless this simplification leads to confusion. By definition,  $g_i \in [0, k_i]$ . Then the probability  $p_i(K, s)$  can be written as

$$p_i(K, s) = \begin{cases} \eta^{g_i}, & \text{if } s_i = b, \\ \xi\eta^{g_i}, & \text{otherwise.} \end{cases}$$

Let us analyze the incentives that investors in bank  $i$  have when choosing either the safe asset or the gambling one (for a given financial network  $K$  chosen at  $t = 0$ ). Investors

will place the bank's resources in the safe asset whenever the possible bank capital loss incurred in gambling is higher than the private benefits. Then for given  $f(k_i)$  and  $s_{-i}$ , investors in bank  $i$  will invest in the safe asset if and only if

$$\eta^{g_i} f(k_i) R x_i \geq \xi \eta^{g_i} f(k_i) R x_i + B,$$

which implies

$$x_i \geq \frac{B}{(1 - \xi) \eta^{g_i} f(k_i) R} \equiv I^*(k_i, g_i, \xi, \eta).$$

Clearly, banks with a relatively low level of bank capital have incentive to invest in the gambling asset while relatively high capitalized banks do not have this incentive.

### 3 Constrained First-Best Network

Let us define an allocation as a vector  $(K, x, s)$ , where:  $x \in \mathcal{X}$ ,  $K \in \mathcal{K}$ , and  $s \in \mathcal{S}$ . An allocation thus specifies a reallocation of initial endowments of capital  $x$ , a network  $K$  and an investment decision  $s$  for each bank. An allocation  $(K, x, s)$  is an *Investment Nash Equilibrium* (INE) for a given economy  $(N, e)$ , with  $x = (x_i)_{i \in N}$ , if

$$m_i(K, x_i, s) \geq m_i(K, x_i, (s_{-i}, \tilde{s}_i)) \quad \text{for all } i \in N,$$

with  $\tilde{s}_i \in \{b, g\}$ . In other words, an allocation is an INE for a given economy if taking the financial network and capital as given there are no unilateral profitable deviations in the choice of the investment project. Note that an allocation  $(K, x, s)$  is an INE for a given economy if and only if for all  $i \in N$

$$s_i = \begin{cases} b, & \text{if } x_i \geq I^*(k_i, g_i, \xi, \eta), \\ g, & \text{otherwise.} \end{cases}$$

The constrained first-best solution is characterized by the social planner problem, which is defined as follows.

**Definition 1** *Given an economy  $(N, e)$ , an allocation  $(K^*, x^*, s^*)$  is a constrained first-best (CFB) if it maximizes*

$$\sum_{i \in N} [M_i(K, s) + m_i(K, x_i, s)] \tag{4}$$

subject to

$$x_i \geq 0 \text{ for all } i \in N, \quad (5)$$

$$\sum_{i \in N} x_i = \sum_{i \in N} e_i \equiv E, \quad (6)$$

$$s_i = \begin{cases} b & \text{if } x_i \geq I^*(k_i, g_i, \xi, \eta) \\ g & \text{otherwise,} \end{cases} \text{ for all } i \in N, \quad (7)$$

$$m_i(K, x, s) \geq \max\{Re_i, \xi Re_i + B\} \text{ for all } i \in N, \quad (8)$$

$$p_i(K, s)f(k_i)R \geq 1. \quad (9)$$

The objective function of the social planner is to maximize the expected return of the agents in the economy (i.e., depositors and investors). We are assuming that the planner is able to (i) transfer the initial endowments of capital across banks, (ii) fix a financial network and (iii) suggest investment plans to the banks. We allow banks to unilaterally deviate from investment decisions (constraint 7). This restricts the social planner problem in a way that moral hazard has to be taken into account. Finally, we also impose the participation constraints for investors (constraint 8) and depositors (constraint 9). We will refer to  $x^*$  as an optimal capital allocation,  $K^*$  as the optimal financial network and  $s^*$  as the optimal investment decision. We first establish the existence of a CFB and a sufficient condition to have the unconstrained (first-best) network.

**Proposition 1** *A CFB allocation always exists. Furthermore, assume that  $E \geq \frac{n-1}{\rho-1} \frac{\rho B}{(1-\xi)R}$ . Then any CFB yields a unique network structure  $K^*$  such that  $K_i^* = N \setminus \{i\}$  for all  $i$  and a unique strategy profile  $s^*$  such that  $s_i^* = b$  for all  $i$ .*

Note that the constraints in the planner's problem, given a network  $K$  and a strategy set  $s$ , do not define a compact set on  $R^n$  because of constraint (7). Nevertheless, we can modify the planner's problem such that the constraints define a compact set on  $R^n$  for any  $K$  and any  $s$  considered. The solution to the modified maximization problem exists and it turns out to be also the solution of the planner's problem. Proposition 1 also establishes a sufficient condition for a CFB to be equal to the first-best, that is, the network in which all banks are connected and are choosing the safe project. This condition assures that there is enough bank capital in the economy so that the first-best is an INE and all participation constraints are satisfied. Intuitively, when bank capital is abundant in the economy the planner can achieve the first-best, avoiding the moral hazard problem. However, in a world where bank capital is scarce the planner problem is a constrained one.

We now characterize the optimal distribution of bank capital made by the planner in the CFB allocation. We use the following notation about transfers that are payoff-equivalent

to the payoff in autarky. Let  $m_i^A = \max\{Re_i, \xi Re_i + B\}$  for each  $i$ . For a given network structure  $K$  and a strategy profile  $s$ , let  $x_i^A(K, s)$  be such that  $m_i(K, x_i^A(K, s), s) = m_i^A$ . That is,  $x_i^A(K, s)$  is a reallocation of bank capital that makes bank  $i$  indifferent between participating or not in the network  $K$  with strategy profile  $s$ . Note that the reallocation of bank capital  $x_i^A(K, s)$  is unique given  $(K, s)$ .

**Proposition 2** *Let  $(K^*, x^*, s^*)$  be a CFB for a given economy  $(N, e)$ . Then,*

1. *For every bank  $i$  such that  $s_i^* = g$ : if there exists another bank  $j$  with  $k_j^* \geq k_i^*$  such that either  $s_j^* = b$  and  $g_j \leq g_i$ , or  $s_j^* = g$  and  $g_j < g_i$ , then  $x_i^* = x_i^A(K^*, s^*)$ .*
2. *For every bank  $i$  such that  $s_i^* = b$  and  $g_i > 0$ : if there exists another bank  $j$  with  $k_j^* \geq k_i^*$  such that either  $s_j^* = b$  and  $g_j < g_i$  or  $s_j^* = g$  with  $g_j < g_i - 1$ , then  $x_i^* = \max\{x_i^A(K^*, s^*), I^*(k_i, g_i, \xi, \eta)\}$ .*

Proposition 2 states that once the participation and incentive constraints are satisfied, the planner will distribute the extra amount of bank capital into the nodes that yield a higher expected payoff, either because they are better connected or because they face a smaller risk of bankruptcy. Note that there might be multiple ways of allocating the initial endowment of capital once the incentive and participation constraints are satisfied. This occurs when there are two or more banks with the same expected payoff. This is illustrated clearly in Example 2 in Section 3.1.

We then establish the shape of the CFB network, which turns out to be characterized by a core-periphery structure. In this structure, the *core* banks choose the safe asset and are all connected to each other. The *periphery* banks choose the gambling asset and they can eventually be connected to some core banks and/or some periphery banks depending on the value of the parameters.

**Proposition 3** *Let  $(K^*, x^*, s^*)$  be a CFB for a given economy  $(N, e)$ . Then, for every pair of banks  $i$  and  $j$  such that  $s_i^* = s_j^* = b$  we have that  $i \in K_j^*$  and  $j \in K_i^*$ .*

The intuition behind the optimality of the core-periphery structure is as follows. When two banks are investing in the safe project, it is always better to have them connected than not connected. This is true since one more neighbor always increases the possibility of liquidity coinsurance. Given that both banks are choosing the safe project, linking them together does not impose any additional risk for them. The same link does not induce them to switch investment decision from the safe to the gambling project. Indeed, if a

bank has enough bank capital to choose the safe project in a given financial network and it prefers this allocation to autarky, then the same bank capital will be sufficient to avoid the gambling behavior if the bank has one more neighbor that invests in the safe project.

The next proposition determines the size of the core in an optimal network. Given an allocation  $(K, x, s)$ , we denote with  $C(K, x, s)$  the set of banks that choose the safe asset. Note that if  $(K, x)$  is equal to  $(\emptyset, e)$  then all the banks are in autarky. It is easy to see that there is a unique INE in autarky, denoted  $(\emptyset, e, s^A)$ , such that for any bank  $i$

$$s_i^A = \begin{cases} b, & \text{if } e_i \geq I^*(0, \eta, 0) = \frac{B}{(1-\xi)R}, \\ g, & \text{otherwise.} \end{cases} \quad (10)$$

**Proposition 4** *Let  $(K^*, x^*, s^*)$  be a CFB allocation. Then,  $C(\emptyset, e, s^A) \subseteq C(K^*, x^*, s^*)$ .*

Proposition 4 implies that the size of the core in a CFB allocation can only increase as compared to the number of safe banks in autarky. More importantly, it also states that the core in a CFB allocation has to include the banks that invest in the safe project in autarky. The intuition is as follows. A bank that invests in the safe project in autarky is a bank with a relative high initial endowment of bank capital. But the higher the initial endowment of bank capital, the higher the capital the planner has to allocate to satisfy the investors' participation constraint of that bank. Precisely, the minimum bank capital the planner has to offer to make investors participate in the optimal network is sufficiently high to induce investors to choose the safe project in an INE for that network. Since the optimal allocation is an INE, Proposition 4 follows.

We are now in a position to characterize the number of connections that each bank should have in a CFB network. The number of optimal links will depend on the relationship between counterparty risk  $\eta$  and the benefits of coinsurance captured by the values of the function  $f(k)$ .

**Proposition 5** *Let  $\underline{k}(\eta)$  be the highest  $k \in \{1, \dots, n-1\}$  such that  $\eta \geq \frac{f(k-1)}{f(k)}$  and let  $\bar{k}(\eta)$  be the lowest  $k \in \{0, 1, \dots, n-2\}$  such that  $\eta < \frac{f(k)}{f(k+1)}$ . Then:*

1. *If  $(K^*, x^*, s^*)$  is a CFB allocation, then,  $k_i^* < \underline{k}(\eta)$  for some  $i$  implies that either  $s_i^* = s$  and  $K_i^* = N \setminus \{i\}$  or  $s_i^* = g$  and  $k_j^* \geq \underline{k}(\eta)$  for all  $j \notin K_i^*$ .*
2. *If  $(K^*, x^*, s^*)$  is a CFB allocation, then,  $k_i^* > \bar{k}(\eta)$  and  $s_i^* = g$  implies that for all  $j \in G_i$ :*

$$(a) \quad k_j \leq \bar{k}(\eta);$$

- (b) *there is no other bank  $r$  with  $k_r^* < \bar{k}(\eta)$  and  $r \notin K_j$ ;*
- (c) *for any other bank  $r$  with  $k_r^* > \bar{k}(\eta)$ ,  $s_r^* = g$  there is no bank  $z \in G_r$  such that  $z \notin K_j$ .*

The first statement of Proposition 5 establishes that in any CFB allocation two unconnected banks cannot have less than  $\underline{k}(\eta)$  links. That is, whenever a bank has less than  $\underline{k}(\eta)$  links then all the banks that are not already connected to it have at least  $\underline{k}(\eta)$  links already. Otherwise, the planner would connect two banks with less than  $\underline{k}(\eta)$  links, getting a higher total expected payoff. To grasp the intuition consider the following example. Assume that  $\eta \geq \frac{f(1)}{f(2)} \geq \frac{1}{f(1)}$  and that for any  $k \geq 2$  we have  $\eta < \frac{f(k)}{f(k+1)}$ . Then  $\underline{k}(\eta) = 2$ . Note that by concavity of the function  $f(k)$  the ratio  $\frac{f(k)}{f(k+1)}$  is increasing on  $k$ . This means that counterparty risk is such that adding a gambling bank is convenient whenever its current links are less than two. Therefore, connecting two banks (no matter their investment strategy) with less than two links always yields a higher total expected payoff than leaving them unconnected.

Statement 2a of Proposition 5 establishes that two periphery banks cannot have more than  $\bar{k}(\eta)$  links and be either directly connected or connected to other (possibly disconnected) periphery banks. For example, assume again that  $\eta \geq \frac{f(1)}{f(2)} \geq \frac{1}{f(1)}$ , and that for any  $k \geq 2$  we have  $\eta < \frac{f(k)}{f(k+1)}$ . Then  $\bar{k}(\eta) = 2$ . Hence, the counterparty risk outweighs the advantages of connecting a gambling bank when the potential neighbor already has two links. Statement 2a says that when both banks are investing in the gambling project no link should exist between them if they already have two links. Therefore, disconnecting two gambling banks when both have more than two links yields a higher expected total payoff than leaving them connected.

Following the previous example, statement 2b of Proposition 5 affirms that if two gambling banks  $i$  and  $j$  are connected, but bank  $i$  has more than two links, the planner can increase the total expected payoff by severing their connection and connect the less connected bank  $j$  to another gambling bank  $r$  that has less than two links. Finally, statement 2c of Proposition 5 says that if two gambling banks, called  $i$  and  $r$ , both with more than two links, have gambling neighbors,  $j$  and  $z$ , that are not linked with each other, the planner can increase the total expected payoff by connecting  $j$  and  $z$  while disconnecting them from  $i$  and  $r$ , respectively.

The next corollary characterizes the CFB allocation further. It determines the two extreme core-periphery structures: 1) the complete network structure, in which all banks are connected no matter their investment strategy, and 2) the empty periphery structure, where only safe banks are connected among them and the gambling ones are disconnected

among each other (but they could be connected to safe banks). The corollary also characterizes the level of connectivity for intermediate values of  $\eta$ .

**Corollary 1** *Let  $(K^*, x^*, s^*)$  be a CFB for a given economy  $(N, e)$ .*

- *If  $\underline{k}(\eta) = n - 1$  then  $K_i^* = N \setminus \{i\}$  for all  $i \in N$ .*
- *If  $\bar{k}(\eta) = 0$  then  $g_i^* = 0$  for all  $i$  such that  $s_i^* = g$ .*
- *If  $\underline{k}(\eta) \neq n - 1$  and  $\bar{k}(\eta) \neq 0$  then  $\underline{k}(\eta) = \bar{k}(\eta)$ .*

First, Corollary 1 states that for  $\eta \geq \frac{f(n-2)}{f(n-1)}$  the CFB network is the complete one. The intuition is that counterparty risk is small with respect to the benefits provided by liquidity coinsurance. Hence, connecting until the last gambling bank in the network increases the expected total payoff. Second, Corollary 1 says that when  $\eta < \frac{1}{f(1)}$  the CFB network is the one where periphery banks can be only connected to core banks. In this case counterparty risk is so high that it is not optimal to link two gambling banks. Indeed, it is not optimal to connect two gambling banks whenever

$$\xi \eta^{g_i} f(k_i) > \xi \eta^{g_i+1} f(k_i + 1) \implies f(k_i) > \eta f(k_i + 1).$$

Since  $f(k_i)$  is concave, if the condition

$$\eta < \frac{f(0)}{f(1)} = \frac{1}{f(1)}$$

holds, then it is never optimal to connect two gambling banks. Third, for intermediate values of  $\eta$  Corollary 1 establishes that  $\underline{k}(\eta) = \bar{k}(\eta)$ , and we define  $k(\eta) \equiv \underline{k}(\eta) = \bar{k}(\eta)$ . Proposition 5 states that in a CFB allocation and for the intermediate values of  $\eta$ : (i) Two non directly connected banks cannot have less than  $k(\eta)$  links, and (ii) two gambling banks cannot be directly connected and both have more than  $k(\eta)$  links.

Note that Corollary 1 does not preclude the possibility for the planner to find it optimal to link a gambling bank to a safe bank. The following Proposition establishes a sufficient condition under which the planner does not find optimal to make such a connection.

**Proposition 6** *Let  $(K^*, x^*, s^*)$  be a CFB for a given economy  $(N, e)$ . Then, if  $\eta < \frac{1}{\rho[1+n\rho]}$  then  $K_i^* = \emptyset$  for all  $i$  with  $s_i^* = g$ .*

The intuition is as follows. The planner finds optimal to connect a gambling bank with a safe bank if the expected coinsurance gains for the former are high enough to outweigh the counterparty risk taken by the latter. Clearly the higher the counterparty risk is (i.e., the

smaller is  $\eta$ ) the larger the risk is for the safe bank to be connected to the gambling bank. At some point, counterparty risk becomes so high that the expected gains for the gambling bank cannot outweigh the risk taken by the safe bank. Note that we have  $\eta < \frac{1}{\rho[1+n\rho]} < \frac{1}{\rho} < \frac{1}{f(1)}$ . Then, whenever it is not optimal to link a safe bank with a gambling bank, it is also not optimal to connect two gambling banks.

### 3.1 Examples of CFB Networks

In this Section we provide two examples. In the first example we illustrate the meaning of the cutoff value  $k(\eta)$  in Proposition 5. In the second example, we discuss the relationship between fragility and network connectivity of CFB networks.

**Example 1.** Assume there are 10 banks and  $k(\eta) = \underline{k}(\eta) = \bar{k}(\eta) = 1$ . Consider two networks as depicted in Figures 1 and 2 below.

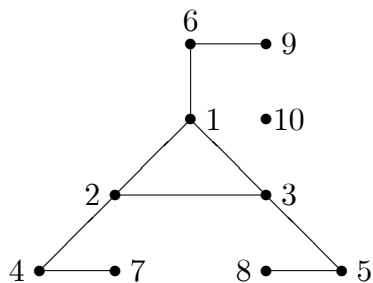


FIGURE 1

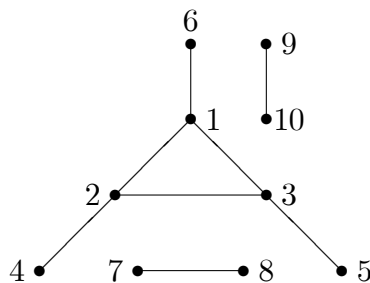


FIGURE 2

In Figure 1 banks 1, 2, and 3 have three links and, since they are connected, they cannot be gambling. Then if the network in Figure 1 is a CFB network it must be the case that banks 1, 2, and 3 are choosing the safe project, representing therefore the core of the CFB network. The other banks are gambling. However, banks 4, 5 and 6 have two links and bank 10 has zero links. Therefore, the network represented in Figure 1 cannot be a CFB network because  $k(\eta) = 1$ , which means that it is not optimal to have an additional gambling neighbor when a bank already has one risky link. The CFB network is as depicted in Figure 2, where all the banks have one risky link.

Finally, note that if we added an eleventh bank investing in the gambling project the CFB network structure could be affected in various ways. Depending on the value of the capital endowments it could be optimal to connect the additional bank to a gambling bank (in which case there would be one bank with more than one risky link) or leave

it disconnected (in which case there would be one bank with less than one risky link). Proposition 5 captures any of such possibilities.

**Example 2.** We determine now the CFB network structures in an economy with four banks. The four-bank case has been widely used in banking literature to assess the relationship between financial fragility and the network structure among the different banks (see Allen and Gale [3]; Brusco and Castiglionesi, [9]). However, in banking theory the network structure is taken as given. We want to analyze how this relationship is affected when the network is endogenously determined. Since there are four banks the maximum number of neighbors  $k_i$  is 3.

Assume that the benefit of being connected is captured by the function

$$f(k) = \begin{cases} 1, & \text{if } k = 0, \\ \frac{3}{2}, & \text{if } k = 1, \\ \frac{7}{4}, & \text{if } k = 2, \\ \frac{15}{8}, & \text{if } k = 3. \end{cases}$$

To simplify the analysis we assume that  $\xi = \eta$  and  $R = B$ . Moreover, assume that bank capital endowments are:  $e_1 = 5$ ,  $e_2 = e_3 = \frac{15}{7}$  and  $e_4 = \frac{5}{7}$ . The total bank capital available  $E$  is then equal to 10. Table 2 reports the cut-off values of  $I^*(k_i, g_i, \xi, \eta)$  for given  $\eta$  and  $\xi = \eta$ .

**Table 2. Cut-off values of  $I^*(k_i, g_i, \xi, \eta)$**

	$k_i = 0$	$k_i = 1$	$k_i = 2$	$k_i = 3$
$g_i = 0$	$\frac{1}{1-\eta}$	$\frac{2}{3(1-\eta)}$	$\frac{4}{7(1-\eta)}$	$\frac{8}{15(1-\eta)}$
$g_i = 1$	—	$\frac{2}{3\eta(1-\eta)}$	$\frac{4}{7\eta(1-\eta)}$	$\frac{8}{15\eta(1-\eta)}$
$g_i = 2$	—	—	$\frac{4}{7\eta^2(1-\eta)}$	$\frac{8}{15\eta^2(1-\eta)}$
$g_i = 3$	—	—	—	$\frac{8}{15\eta^3(1-\eta)}$

Clearly, the CFB network structure depends both on the total bank capital available  $E$  and on how the parameter  $\eta$  relates to the values of the function  $f(k)$ . If the aggregate bank capital satisfies the condition  $E \geq 4I^*(3, 0, \xi, \eta)$ , then, the CFB network structure coincides with the first-best (i.e., the complete network with all the banks linked together and investing in the safe asset). The condition can be written as

$$10 \geq 4 \times \frac{8}{15(1-\eta)},$$

which implies that if  $\eta \leq \frac{59}{75}$  the complete and safe network structure is incentive compatible. It remains to check whether the participation constraints for each of the four banks can be satisfied.

Assume that  $\xi = \eta = \frac{3}{5} < \frac{59}{75}$  and thus  $I^*(0, 0, \xi, \eta) = \frac{5}{2}$ . Given the bank capital endowments  $e_i$ , only bank 1 chooses the safe project in autarky. The corresponding payoffs in autarky would then be  $m_1^A = 5R$ ,  $m_2^A = m_3^A = \frac{9}{7}R + B = \frac{16}{7}R$  and  $m_4^A = \frac{3}{7}R + B = \frac{10}{7}R$ . The reader might check that the re-allocation of capital able to implement the first-best network has to satisfy the following constraints:

- Participation constraints:  $x_1^* \geq \frac{8}{3}$ ,  $x_i^* \geq \frac{8}{15} \times \frac{16}{7}$  for  $i = 2, 3$  and  $x_4^* \geq \frac{8}{15} \times \frac{10}{7}$ .
- Incentive constraints (INE):  $x_i^* \geq \frac{4}{3}$  for  $i = 1, 2, 3, 4$ .

Among the many allocations that satisfy the participation and incentive constraints above, the planner can choose for example  $x_1^* = 5$  and  $x_2^* = x_3^* = x_4^* = \frac{5}{3}$  which allows the planner to implement the complete first-best network.

If  $\xi = \eta > \frac{59}{75}$  then the first-best allocation is no longer attainable for the planner. Let us explore under which conditions a core with three banks is feasible (and optimal). Figure 3 shows four core-periphery network structures under the assumption that the core is made of three banks. In particular, the two banks represented at the top and the one at the bottom right are investing in the safe project while the one at the bottom left is investing in the gambling project.

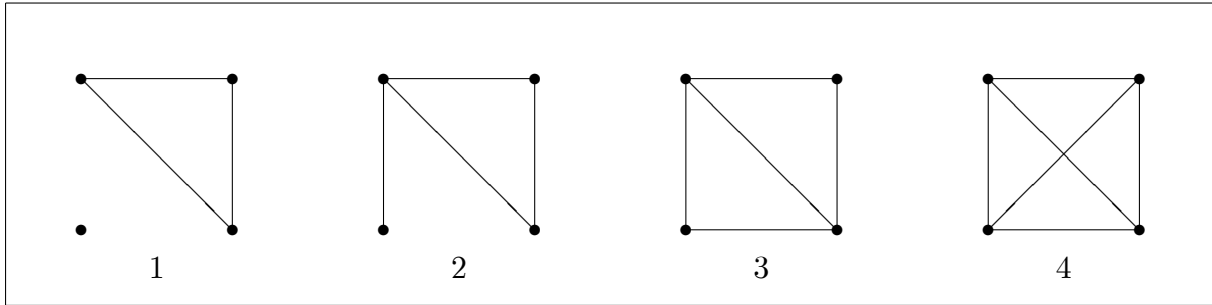


FIGURE 3

Table 3 reports the minimum bank capital that satisfies the INE condition for network structures 1, 2, 3 and 4 in Figure 3. It also reports the minimum bank capital given the cut-off values  $I^*(k_i, g_i, \xi, \eta)$  in Table 2.

**Table 3. Minimum aggregate bank capital for network structures in Figure 3**

Structure	Minimum Capital	Minimum Capital Given Table 2
1	$3I^*(2, 0, \xi, \eta) = E_1$	$E_1 = \frac{12}{7(1-\eta)}$
2	$2I^*(2, 0, \xi, \eta) + I^*(3, 1, \xi, \eta) = E_2$	$E_2 = \frac{120\eta+56}{105\eta(1-\eta)}$
3	$I^*(2, 0, \xi, \eta) + 2I^*(3, 1, \xi, \eta) = E_3$	$E_3 = \frac{60\eta+112}{105\eta(1-\eta)}$
4	$3I^*(3, 1, \xi, \eta) = E_4$	$E_4 = \frac{8}{5\eta(1-\eta)}$

In order for structure 1 to be an INE the aggregate bank capital has to be at least  $E_1$ . The equivalent applies to the other structures. For each network structure with three banks in the core, there are different ways of allocating the aggregate bank capital. The optimal choice depends on how the values of  $\eta f(3)$  and  $f(2)$  are related.

Assume that  $\eta = \frac{4}{5} > \frac{59}{75}$ . This implies that  $\eta f(3) < f(2)$ . That is, a safe bank with three neighbors, with one of them investing in the gambling project, obtains a lower expected payoff than a safe bank with two neighbors both investing in the safe project. Then, structure 1 could be the CFB network. Moreover,  $\xi = \eta = \frac{4}{5}$  implies that  $I^*(0, 0, \xi, \eta) = 5$ . Given the initial endowments of bank capital, only bank 1 chooses the safe project in autarky. The expected payoffs in autarky are then equal to  $m_1^A = 5R$ ,  $m_2^A = m_3^A = \frac{12}{7}R + B = \frac{19}{7}R$  and  $m_4^A = \frac{4}{7}R + B = \frac{11}{7}R$ .

Let us check whether a core with three banks is attainable for the planner. Under the assumption that  $\eta f(3) < f(2)$ , we have that  $E_1 < E_2 < E_3 < E_4$ . Since  $E = E_4$  then all the structures in Figure 3 are incentive compatible. For example, the re-allocation of capital able to implement structure 1 has to satisfy the following constraints:

- Participation constraints:  $x_1^* \geq \frac{20}{7}$ ,  $x_i^* \geq \frac{4}{7} \times \frac{19}{7}$  for  $i = 2, 3$  and  $x_4^* \geq \frac{5}{7}$ .
- Incentive constraints (INE):  $x_i^* \geq \frac{20}{7}$  for  $i = 1, 2, 3$  and  $x_4^* < 5$ .

Any allocation of capital leaving bank 4 with its initial endowment of bank capital and giving at least  $\frac{20}{7}$  of bank capital to each of the other three banks sustains structure 1. There is enough capital in the economy, since  $\frac{20}{7} \times 3 + \frac{5}{7} = \frac{65}{7}$  is smaller than  $E$ . Finally, given that  $\eta < \frac{f(1)}{f(2)}$ , structure 1 is the CFB structure since it yields the highest total expected payoffs.

The main implication is that, given the initial bank capital endowments, the higher is  $\eta$  (i.e., the lower the counterparty risk) and the smaller is the size of the core in a CFB. Consider a higher value of  $\eta$ , for example  $\eta = \frac{6}{7}$ . The first-best network is not feasible since  $I^*(3, 0, \xi, \eta) = \frac{56}{15}$  and  $4I^*(3, 0, \xi, \eta) > E$ . Also with  $\eta = \frac{6}{7}$  we have  $\eta f(3) < f(2)$  and the minimal capital requirements in Table 3 relate as  $E_1 < E_2 < E_3 < E_4$ . Note that  $E < E_1$ ,

therefore when  $\eta = \frac{6}{7}$  the CFB network contains at most two banks in the core that invest in the safe project.

Figure 4 represents some core-periphery structures where the core is made of two banks.<sup>6</sup> The two banks represented at the top are the core ones, while the two banks at the bottom are the periphery ones. The latter banks could be disconnected (as in structures 5 to 8) or connected (as in structures 9 to 12). Moreover, the periphery banks could be unconnected to the core (as in structures 5 and 9) or connected to one or both core banks.

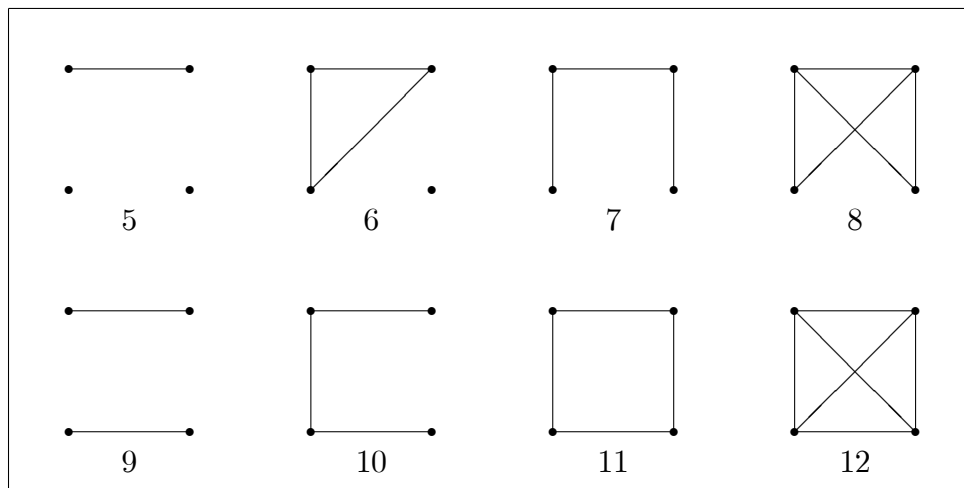


FIGURE 4

According to Corollary 1, if  $\eta > \frac{f(2)}{f(3)} = \frac{14}{15}$  then the CFB structure would be the fully connected one. If bank capital endowments were large enough for sustaining two banks in the core, structure 12 will be the CFB network. We can then derive a first observation on the relationship between fragility and connectivity: The same network structure (in this case, a fully connected one) can imply different degrees of financial fragility. The fully connected (first-best) network has zero probability of collapse since moral hazard was avoided, while structure 12 has a positive probability of meltdown. Indeed, the probability to have a contagious default is either when a gambling bank has a high liquidity shock and the other three banks have a low liquidity shock (event that occurs with probability  $2/16$ ) or when the two gambling banks have the high shock and the two safe banks have a low shock (event that occurs with probability  $1/16$ ). In the former case the failure of the gambling bank brings down the system, in the latter case both gambling banks have to fail

<sup>6</sup>There are fourteen *isomorphic* core-periphery structures for four banks so that exactly two banks form the core.

to cause the meltdown. Then structure 12 has a probability of collapse equal to

$$\frac{2}{16}(1 - \bar{\eta}) + \frac{1}{16}(1 - \bar{\eta})^2 \quad (11)$$

with  $\bar{\eta} > \frac{14}{15}$ . Even if the latter represents a very small probability, it still implies that the CFB network is exposed to financial fragility. Two identical network structures can imply different degrees of financial fragility due to the corresponding investment decisions.

Let us analyze when structure 11 is the CFB network. When  $\eta = \frac{6}{7} < \frac{14}{15}$  the optimal network structure contains at most two banks in the core. However, with the initial endowments of capital, the planner cannot satisfy all incentive and participation constraints for  $\frac{f(1)}{f(2)} \leq \eta < \frac{f(2)}{f(3)}$ . Assume then that the initial bank capital endowments are:  $e_1 = 5$ ,  $e_2 = 4$  and  $e_3 = e_4 = \frac{1}{2}$ . In a CFB allocation the planner could assign  $x_1 = x_2 = \frac{14}{3}$  and  $x_3 = x_4 = \frac{1}{3}$ , connect the banks as in structure 11 and recommend banks 1 and 2 to invest in the safe project and banks 3 and 4 to invest in the gambling one.

- Incentive constraints:  $I^*(2, 1, \frac{6}{7}, \frac{6}{7}) = \frac{14}{3}$ , therefore banks 1 and 2 choosing the safe project and banks 3 and 4 investing in the gambling project is an INE.
- Participation constraints: Since  $I^*(0, 0, \xi, \eta) = 7 > 5$  all banks choose the gambling project in autarky. The expected payoffs in autarky are therefore as follows:

$$m_1^A = \frac{6}{7}5R + B = \frac{30}{7}R + B = \frac{37}{7}R \quad (12)$$

$$m_2^A = \frac{6}{7}4R + B = \frac{24}{7}R + B = \frac{31}{7}R \quad (13)$$

$$m_3^A = m_4^A = \frac{6}{7}\frac{1}{2}R + B = \frac{3}{7}R + B = \frac{10}{7}R \quad (14)$$

Furthermore, recall that the planner needs to satisfy the depositors' participation constraints  $\xi\eta f(2)R \geq 1$  and  $\eta^2 f(2)R \geq 1$ . This means that  $\frac{9}{7}R \geq 1$  which is true for  $R > 1$ . Note that structure 12 would not be feasible in this case: The planner needs to transfer to banks 1 and 2 at least  $\frac{686}{135}$  of bank capital to satisfy their incentive constraints. This amount is more than the total bank capital  $E$  in the economy.

We can finally compare financial fragility among CFB networks that are characterized by different degrees of connectivity. This is not so straightforward since they are characterized also by different probability of default. For example, let us compare structure 11 and structure 12. Notice that structure 11 will collapse when the two gambling banks receive the high shock and the two safe banks receive the low shock (event that occurs with probability 1/16), and the two gambling banks have to fail. Then, while the probability of failure of the entire system in structure 12 is given by (11), the same probability in

structure 11 is given by  $\frac{1}{16}(1 - \underline{\eta})^2$  with  $\underline{\eta} < \frac{14}{15}$ . Accordingly, financial fragility is higher in structure 12 if

$$\frac{2}{16}(1 - \bar{\eta}) + \frac{1}{16}(1 - \bar{\eta})^2 > \frac{1}{16}(1 - \underline{\eta})^2$$

Otherwise, structure 11 is characterized by higher financial fragility. For example, if  $\underline{\eta} = \bar{\eta} = \frac{14}{15}$  then structure 12 is more fragile. If, instead,  $\bar{\eta} = 0.99$  and  $\underline{\eta} = 7/6$  then structure 11 is more fragile. Then, a higher connectivity of the financial network does not necessarily increase or decrease financial fragility. The relationship between network connectivity and fragility can go either way in a CFB financial network.

## 4 Financial Network Formation

In this section we study the decentralized financial network formation. The scope is to analyze what are the incentives of the banks, if any, to form a CFB network. We assume first that no bank capital transfers are allowed at  $t = 1$ . Then the banks, given their endowment of bank capital, choose the network structure anticipating the INE played in that network. Once the efficiency properties are characterized, we analyze the effects of bank capital transfers.

### 4.1 Decentralized Networks without Transfers

For a given economy  $(N, e)$ , an allocation without transfers  $(K, e, s)$  is an INE if taking the financial network and bank capital as given there are no unilateral profitable deviations in the investment choice. We define an *economy without transfers* an economy  $(N, e)$  in which the sequence of events shown in Table 1 does not include bank capital transfers across banks. Then, given the sequence of events in Table 1, we solve the model backwards from  $t = 2$  until  $t = 0$ .

Without bank capital transfers the network formation game is basically a static one, that is banks choose simultaneously to whom they want to connect. Accordingly, we adopt a static equilibrium notion that is an adapted version of *pairwise stability* introduced by Jackson and Wolinsky [19]. Given a network  $K$ , we can define a new network  $K \cup ij$  resulting from adding a link joining banks  $i$  and  $j$  to the existing network  $K$ . Formally,  $K \cup ij = (\tilde{K}_1, \dots, \tilde{K}_n)$  such that  $\tilde{K}_i = K_i \cup \{j\}$ ,  $\tilde{K}_j = K_j \cup \{i\}$  and  $\tilde{K}_r = K_r$  for all  $r \neq i, j$ . On the contrary, for any two banks  $i$  and  $j$  connected in  $K$ , let  $K \setminus ij$  denotes the resulting network from severing the link joining banks  $i$  and  $j$  from  $K$ . Formally,  $K \setminus ij = (\tilde{K}_1, \dots, \tilde{K}_n)$  such that  $\tilde{K}_i = K_i \setminus \{j\}$ ,  $\tilde{K}_j = K_j \setminus \{i\}$  and  $\tilde{K}_r = K_r$  for all  $r \neq i, j$ .

**Definition 2** An allocation without transfers  $(K, e, s)$  is pairwise stable (PSWT) if the following holds:

1. For all  $i$  and  $j$  directly connected in  $K$ :  $m_i(K, e, s) \geq m_i(K \setminus ij, e, \tilde{s})$  and  $m_j(K, e, s) \geq m_j(K \setminus ij, e, \tilde{s})$  for all allocations  $(K \setminus ij, e, \tilde{s})$  that are INE.
2. For all  $i$  and  $j$  not directly connected in  $K$ : if there is an INE  $(K \cup ij, e, \tilde{s})$  such that  $m_i(K, e, s) < m_i(K \cup ij, e, \tilde{s})$ , then  $m_j(K, e, s) > m_j(K \cup ij, e, \tilde{s})$ .

The definition of PSWT captures two ideas that derive from the notion of pairwise stability. The first idea refers to the network internal stability: No pair of banks directly connected in the current financial network individually gain from severing their financial link. This implies that any of the two banks could sever the link unilaterally. The second idea establishes the network external stability: If one bank could gain from creating a link with another bank, it has to be that the other bank cannot gain from that link. This implies that both banks have to agree in order to create a new link. Note that if one bank strictly gains with the creation of one link and the other bank is indifferent, it is assumed that the link is formed.

**Definition 3** An allocation without transfers  $(K, e, s)$  is a decentralized equilibrium (DEWT) if it is INE and PSWT.

The following proposition describes the set of decentralized equilibria for a given economy without transfers  $(N, e)$ .

**Proposition 7** Assume  $(N, e)$  define an economy without transfers. Then, a DEWT is a core-periphery structure, i.e., if  $(K^e, e, s^e)$  is a DEWT, then, for every pair of banks  $i$  and  $j$  such that  $s_i^e = s_j^e = b$ , we have that  $i \in K_j^e$  and  $j \in K_i^e$ .

The intuiting of the proposition is as follows. On one hand, a bank agrees to be connected to any neighbor that is choosing the safe project since this decision entails no counterparty risk. On the other hand, if a bank invests in the safe project any other bank would like to be connected to it for the same reason. Since links are expected to be beneficial for both participating banks, two banks choosing the safe project will be connected. Therefore a core-periphery structure emerges where all the banks that choose the safe project are connected among themselves. The connectivity of banks choosing the gambling project (the low capitalized banks) depends on how the parameters of the model relate. We are going to see if the conditions in the decentralized network differ from those in the CFB network.

Recall that  $\underline{k}(\eta)$  is the highest  $k \in \{1, \dots, n-1\}$  such that  $\eta \geq \frac{f(k-1)}{f(k)}$  and that  $\bar{k}(\eta)$  be the lowest  $k \in \{0, 1, \dots, n-2\}$  such that  $\eta < \frac{f(k)}{f(k+1)}$ . By Corollary 1, if  $\underline{k}(\eta) \neq n-1$  and  $\bar{k}(\eta) \neq 0$  then  $\underline{k}(\eta) = \bar{k}(\eta) = k(\eta)$ . We have the following

**Proposition 8** *Let  $\underline{k}(\eta)$  and  $\bar{k}(\eta)$  as defined above. Then:*

1. *If  $(K^e, e, s^e)$  is a DEWT allocation, then,  $k_i^e < \underline{k}(\eta)$  for some  $i$  implies that either  $s_i^e = b$  and  $K_i^e = N \setminus \{i\}$  or  $s_i^e = g$  and  $k_j^e \geq \underline{k}(\eta)$  for all  $j \notin K_i^e$ .*
2. *If  $(K^e, e, s^e)$  is a DEWT allocation, then,  $k_i^e > \bar{k}(\eta)$  only if  $G_i = \emptyset$ .*

Statement 1 in Proposition 8 establishes a condition on the creation of links: Pairwise stability states that a link is formed if the two participants agree. If both participants agree to form a link, then the total expected payoff in the economy has to be higher with the link than without it. Note that the expected total marginal payoff of an additional link is at least equal to the sum of the expected marginal payoffs to each of the participants. The total expected marginal payoff of an additional link could be higher than the sum of the expected payoff to the two participants. By adding the link, at least one of the two participants switches from the gambling asset to the safe asset. Statement 2 in Proposition 8 says that no bank is holding too many risky links. Otherwise, a bank could unilaterally break them increasing its expected payoff.

Let us now compare the conditions in the CFB network with those in the decentralized network. That is, let us compare Proposition 8 with Proposition 5. Statement 2 in Proposition 5 also establishes that no bank in a CFB allocation has too many risky links according to  $\bar{k}(\eta)$ . If  $G_i = \emptyset$  then the conditions for  $k_i^* > \bar{k}(\eta)$  in statement 2 of Proposition 5 are satisfied. However, Proposition 5 allows for safe banks to have more risky links with respect to the decentralized network. In particular, in the decentralized network a safe bank will delete a link with a gambling bank if counterparty risk is too high. On the contrary, the planner can find it convenient to link the safe bank to a gambling one if the expected gains for the latter outweigh the expected losses for the former. For example, consider the CFB network in Figure 2. That network is not pairwise stable since banks 1, 2 and 3 will gain by unilaterally severing the link with banks 4, 5, and 6, respectively. The reason being that they already have more than one link, and  $k(\eta) = 1$  implies that  $\eta f(3) < f(2)$ .

The next Corollary states the results for the two extreme values of  $k(\eta)$ .

**Corollary 2** *Let  $(K^e, e, s^e)$  be a DEWT for a given economy without transfers  $(N, e)$ .*

- If  $\underline{k}(\eta) = n - 1$  then  $K_i^e = N \setminus \{i\}$  for all  $i \in N$ .
- If  $\bar{k}(\eta) = 0$  then  $g_i^e = 0$  for all  $i$ .

If  $(N, e)$  define an economy without transfers, and  $\eta \geq \frac{f(n-2)}{f(n-1)}$  then the only network structure for any DEWT is the complete network structure. This result states formally the idea that when counter-party risk is sufficiently low, it is always worthwhile to take the risk of being connected to a low capitalized bank in order to obtain the advantages resulting from liquidity coinsurance.

Note that the results stated in Proposition 8 and Corollary 2, even when they establish that the decentralized network structure could be equal to the CFB one, do not imply that investment decisions are the same in both networks. In the DEWT the investment decisions might be suboptimal. However, when  $\eta$  tends to one, investment profiles yield the same total expected amount of money provided they have the same network structure. In other words, as the counterparty risk vanishes, the only factor that determines the expected total payoff in the economy *is* the network structure.

Corollary 2 states that when counter-party risk is high, the decentralized financial network without transfers becomes safe. The reason is that gambling banks are not linked anymore with anybody else. Only banks that are choosing the safe project have connections in the decentralized network. While Corollary 2 establishes that  $\eta < \frac{1}{f(1)}$  is sufficient to observe an empty periphery in the decentralized network, Corollary 1 states that the same condition is not sufficient for the empty periphery to be optimal. As a consequence, gambling banks can be (inefficiently) under-connected in the decentralized network when

$$\eta \in \left[ \frac{1}{\rho(1+n\rho)}, \frac{1}{f(1)} \right].$$

The result we have established tells us that when counter-party risk is high the decentralized network can be inefficiently under-connected. Only the safe banks are connected among themselves. The financial network shows fewer connections than optimal since parties consider engaging in bilateral agreement too risky. The liquidity insurance mechanism provided by the financial network is not working efficiently when counterparty risk is high, given that safe banks do not internalize the overall benefit of a higher liquidity coinsurance.<sup>7</sup>

The previous result could be due to the absence of a mechanism (like bank capital transfers) that could allow safe banks to internalize the network externality. The question

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<sup>7</sup>Clearly, since counterparty risk  $\eta$  is not the only parameter in determining the shape of a CFB network (bank capital endowments are important as well), the decentralized network is ‘underconnected’ whenever the more connected network is feasible given the initial capital endowments.

we are addressing then is the following: If we allow for bank capital transfers, does the decentralized network mimic the CFB network?

## 4.2 Decentralized Networks with Transfers

We introduce now a sequential-move game in which banks can transfer bank capital to each other. In the sequence of events in Table 1, transfers are made after the network structure is chosen but before the investment decisions take place. The main role of the bank capital transfers is then to prevent other banks from choosing the gambling project. This means that transfers are not contingent on creating or destroying a link. A bank making a positive transfer instead wants to induce the desired behavior of its neighbors in the investment stage, reducing the risk associated with its links.

As in the previous section we solve the model backwards. We analyze the INE given the transfers and the network. Then, given the network, we solve for the transfers anticipating the INE played in the last stage. Finally, we characterize the network anticipating the transfers made and the INE played in the following stages. As mentioned before, our concept of INE comes from a simultaneous-move game in which INE might not be unique. This can be problematic in a decentralized context where there is no coordination device.<sup>8</sup> However, we abstract from possible inefficiencies due to coordination failure in the investment decision. Our focus is on whether bank capital transfers can solve for the inefficiencies in the process of network formation.

To address this issue, we consider a sequential-move transfer and investment decisions process with perfect information, defining the following rule of order. We rank banks according to their bank capital endowment, and we start the order of transfers from the highest to the lowest capital endowed bank. Once the bank endowed with the lowest capital has taken his transfer decision, we move on to the investment stage. In this stage, the bank with the highest capital (which now might be different from the bank with the highest capital endowment because of the transfers) decides the type of investment, and the other banks follow according to their level of bank capital. Solving backwards will allow us to select one INE and one profile of transfers for a given network, that will be a Sub-game Perfect Equilibrium in the transfer and investment game. Once the transfer and investment profile are uniquely determined in equilibrium, we can apply pairwise stability to the network formation process.

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<sup>8</sup>Recall that for the decentralized case where no transfers are allowed, the definition of PSWT does not preclude the possibility of coordination failure, that is the possibility of encountering more than one INE given network  $K$  and the initial endowment  $e$ .

The following proposition states that the resulting INE, selected in the continuation equilibrium after transfers, solve for the coordination problems that appear in the simultaneous-move game, independently of the rule of order.

**Proposition 9** *Let  $K$  be a given network and  $x$  be a vector of bank capital reallocation such that there are multiple INE following  $(K, x)$ . Let  $Q(s, s') \subseteq N$  be the group of banks for which investment decisions change from the gambling to the safe project in two INE  $s, s'$  following  $(K, x)$ . Then the backward induction argument in the sequential-move investment game does not select the INE where agents in  $Q$  choose the gambling project, independently of the rule of order.*

The result in Proposition 9 states that with sequential investment decisions any coordination problem among banks is solved. This means that if banks arrive at the investment stage after choosing the optimal network and having done the optimal transfers, the decentralized financial network coincides with the CFB network. We explore now whether bank capital transfers in a sequential move game lead to a CFB allocation, assuming that INE decisions in the last stage are optimal.

**Proposition 10** *There exists a  $\bar{\xi} \in [0, 1]$  such that if  $\xi > \bar{\xi}$  then there are no transfers in the sequential-move transfer game that can induce a gambling bank to switch its investment decision.*

The intuition is the following. When the default of the gambling bank is sufficiently low (or  $\xi$  is sufficiently high), which implies that counterparty risk is even lower (since  $\eta > \xi$ ), no bank capital transfers occur. Counterparty risk is not high enough, accordingly the amount of bank capital that has to be transferred becomes too costly. Then, whenever  $\xi > \bar{\xi}$ , if the decentralized network is not a CFB it will remain an inefficient network even when transfers are allowed. Moreover, if the decentralized network structure coincides with the CFB network structure, the investment strategies could differ from the CFB. In general, we have higher fragility in the decentralized network since transfers cannot correct in general the gambling behavior of some banks.

What happens when the gambling project becomes riskier and so also counterparty risk is higher? In order to analyze this case, let us formalize the network formation process further. Solving backwards the continuation equilibrium we obtain a unique solution in capital allocation  $x^T(K)$  and in investment decision  $s^I(K, x^T(K))$ . Therefore, we indicate the expected payoff for bank  $i$ 's investors at a given network  $K$  as  $v_i(K) \equiv m_i(K, x^*(K), s^I(K, x^T(K)))$ .

**Definition 4** An allocation with transfers  $(K, x, s)$  is pairwise stable (PST) if the following holds:

1. For all  $i$  and  $j$  directly connected in  $K$ :  $v_i(K) \geq v_i(K \setminus ij)$  and  $v_j(K) \geq v_j(K \setminus ij)$ .
2. For all  $i$  and  $j$  not directly connected in  $K$ : if  $v_i(K) < v_j(K \setminus ij)$ , then  $v_j(K) > v_j(K \setminus ij)$ .

The definition of PST is equivalent to pairwise stability defined before. We now provide a counter example in which the CFB network is not PST even when transfers are correct for each given network structure. This shows that CFB allocations cannot be implemented in our decentralized environment with transfers.

**Example 2 (continued).** In the four-bank example we maintain all the previous assumptions. In particular, bank capital endowments are:  $e_1 = 5$ ,  $e_2 = e_3 = \frac{15}{7}$  and  $e_4 = \frac{5}{7}$ . We focus the attention on the case in which  $\xi = \eta = \frac{4}{5}$ .<sup>9</sup> As shown before, the CFB network is given in this case by structure 1 in Figure 3, that is, it consists of a core-periphery structure in which the core contains 3 banks: banks 1, 2 and 3 are the core banks and bank 4 is the gambling bank. Recall that the banks in the core obtain at least  $\frac{20}{7}$  dollars of bank capital if the incentive and participation constraints have to be satisfied.

In order to show that the CFB is not pairwise stable with transfers, we compare the payoffs at the CFB network with the one in which a link is added to the CFB network structure. Then, the CFB network structure  $K^*$  lists  $K_1^* = \{2, 3\}$ ,  $K_2^* = \{1, 3\}$ ,  $K_3^* = \{1, 2\}$  and  $K_4^* = \emptyset$ . Let us compute the payoffs at  $K^*$  by fixing the rule of order to be equal to  $(1, 2, 3, 4)$ . We solve backwards, starting with bank 4 with its investment decision.<sup>10</sup>

When bank 4 takes its investment decision, there are several histories  $h$ . Let  $x_i(h)$  be the amount of bank capital available to bank  $i$  after history  $h$ . Since bank 4 is disconnected in network  $K^*$ , bank 4 will choose the safe project if  $x_4(h) \geq I^* (0, 0, \frac{4}{5}, \frac{4}{5}) = 5$  independently of the investment decisions in history  $h$ .

When bank 3 decides its investment project, the decision does not depend on the future choice of bank 4. There are three possibilities according to what happened at history  $h$ : (i)

<sup>9</sup>Notice that, since  $\bar{\xi} = \max\{1 - \frac{B}{\rho RE}, 0\}$ , in this example the cut off value where no transfers will occur is  $\bar{\xi} = \frac{19}{20}$ . So we are considering a case where Proposition 10 does not hold.

<sup>10</sup>For reasons that will become clear during the example, we assume a final round after the four investment decisions, called the ‘revision round’. During this round banks can observe the choices of the other banks and are allowed to change their strategy at a small cost  $\epsilon$ . In this way, we guarantee that after the investment game only INE profiles are selected, for any given history of transfers (even the ones out of equilibrium). This assumption simplifies the analysis but it is not crucial for the result.

If both banks 1 and 2 are choosing the safe project, bank 3 chooses the safe project only if  $x_3(h) \geq \frac{20}{7}$ ; (ii) If one bank between banks 1 and 2 is choosing the gambling project, bank 3 chooses the safe project only if  $x_3(h) \geq \frac{25}{7}$ ; (iii) If both banks 1 and 2 are choosing the gambling project, bank 3 chooses the safe project only if  $x_3(h) \geq \frac{125}{28}$ .

When bank 2 decides its investment project there are six possibilities depending on history  $h$ . Assume that bank 1 chooses the safe project. Then: (i) If  $x_3(h) \geq \frac{25}{7} > \frac{20}{7}$  bank 3 chooses the safe project independently of the choice of bank 2, and therefore bank 2 chooses the safe project only if  $x_2(h) \geq \frac{20}{7}$ ; (ii) If  $\frac{20}{7} \leq x_3(h) < \frac{25}{7}$  bank 3 chooses the safe project only if bank 2 chooses the safe project, while bank 3 will choose the gambling project if bank 2 chooses the gambling project, and therefore bank 2 chooses the safe project only if

$$f(2)Rx_2(h) \geq \eta^2 f(2)Rx_2(h) + B,$$

which implies  $x_2(h) \geq \frac{100}{63}$  and, thanks to the revision round at the end of the game, if  $x_2(h) \geq \frac{20}{7}$  (otherwise bank 2 will change its investment decision in the revision round even if it will choose the safe investment); (iii) If  $x_3(h) < \frac{20}{7}$  bank 3 chooses the gambling project independently of the choice of bank 2, therefore bank 2 chooses the safe project only if  $x_2(h) \geq \frac{25}{7}$ .

Assume that bank 1 chooses the gambling project. Then: (iv) If  $x_3(h) \geq \frac{125}{28} > \frac{25}{7}$  bank 3 chooses the safe project independently of the choice of bank 2, and therefore bank 2 chooses the safe project only if  $x_2(h) \geq \frac{25}{7}$ ; (v) If  $\frac{25}{7} \leq x_3(h) < \frac{125}{28}$  bank 3 chooses the safe project only if bank 2 chooses the safe project, while bank 3 chooses the gambling project if bank 2 does the same, and therefore bank 2 chooses the safe project only if

$$\eta f(2)Rx_2(h) \geq \eta^3 f(2)Rx_2(h) + B,$$

which implies  $x_2(h) \geq \frac{125}{63}$  and if  $x_2(h) \geq \frac{25}{7}$  (otherwise bank 2 will change its investment decision in the revision round if it were to choose the safe investment now); (vi) If  $x_3(h) < \frac{25}{7}$  bank 3 chooses the gambling project independently of the choice of bank 2, and therefore bank 2 chooses the safe project only if  $x_2(h) \geq \frac{125}{28}$ .

In Tables 4 and 5 we summarize the investing decisions of banks 2 and 3, that is  $s_2(h)$  and  $s_3(h)$ , depending on the values of  $x_2(h)$  and  $x_3(h)$ . Table 4 shows the investment decisions when bank 1 chooses the safe project, while Table 5 shows the investment decisions when bank 1 chooses the gambling one.

**Table 4.** Investment decisions  $(s_2(h), s_3(h))$  when bank 1 chooses the safe project

	$x_3(h) < \frac{20}{7}$	$\frac{20}{7} \leq x_3(h) < \frac{25}{7}$	$\frac{25}{7} \leq x_3(h) < \frac{125}{28}$	$x_3(h) \geq \frac{125}{28}$
$x_2(h) < \frac{20}{7}$	$(g, g)$	$(g, g)$	$(g, b)$	$(g, b)$
$\frac{20}{7} \leq x_2(h) < \frac{25}{7}$	$(g, g)$	$(b, b)$	$(b, b)$	$(b, b)$
$\frac{25}{7} \leq x_2(h) < \frac{125}{28}$	$(b, g)$	$(b, b)$	$(b, b)$	$(b, b)$
$x_2(h) \geq \frac{125}{28}$	$(b, g)$	$(b, b)$	$(b, b)$	$(b, b)$

**Table 5.** Investment decisions  $(s_2(h), s_3(h))$  when bank 1 chooses the gambling project

	$x_3(h) < \frac{20}{7}$	$\frac{20}{7} \leq x_3(h) < \frac{25}{7}$	$\frac{25}{7} \leq x_3(h) < \frac{125}{28}$	$x_3(h) \geq \frac{125}{28}$
$x_2(h) < \frac{20}{7}$	$(g, g)$	$(g, g)$	$(g, g)$	$(g, b)$
$\frac{20}{7} \leq x_2(h) < \frac{25}{7}$	$(g, g)$	$(g, g)$	$(g, g)$	$(g, b)$
$\frac{25}{7} \leq x_2(h) < \frac{125}{28}$	$(g, g)$	$(g, g)$	$(b, b)$	$(b, b)$
$x_2(h) \geq \frac{125}{28}$	$(b, g)$	$(b, g)$	$(b, b)$	$(b, b)$

When bank 1 decides its investment project it takes into account what the other 2 banks will do. Given the investment decisions of banks 2 and 3 in Tables 4 and 5, we have nine possible cases for the investment decision of bank 1:

1. Both banks 2 and 3 choose the safe project independently of the choice of bank 1. This happens when  $x_2(h) \geq \frac{25}{7}$  and  $x_3(h) \geq \frac{25}{7}$ . Bank 1 will choose the safe project, even after the revision round, if  $x_1(h) \geq \frac{20}{7}$  resulting in the investment profile  $(s_1(h), s_2(h), s_3(h)) = (b, b, b)$ . Otherwise, bank 1 chooses the gambling project and the investment profile is  $(g, b, b)$ .
2. Both banks 2 and 3 choose the gambling project independently of the choice of bank 1. This happens when  $x_2(h) < \frac{20}{7}$  and  $x_3(h) < \frac{25}{7}$  or when  $\frac{20}{7} \leq x_2(h) < \frac{25}{7}$  and  $x_3(h) < \frac{20}{7}$ . Bank 1 will choose the safe project, even after the revision round, if  $x_1(h) \geq \frac{125}{28}$  resulting in the investment profile  $(b, g, g)$ . Otherwise, the investment profile is  $(g, g, g)$ .
3. Bank 2 chooses the safe project and bank 3 chooses the gambling project independently of the choice of bank 1. This happens when  $x_2(h) \geq \frac{125}{28}$  and  $x_3(h) < \frac{20}{7}$ . Bank 1 will choose the safe project if  $x_1(h) \geq \frac{25}{7}$  resulting in the investment profile  $(b, b, g)$ . Otherwise, the investment profile is  $(g, b, g)$ .
4. Bank 2 chooses the gambling project and bank 3 chooses the safe project independently of the choice of bank 1. This happens when  $x_2(h) < \frac{20}{7}$  and  $x_3(h) \geq \frac{125}{28}$ . Bank 1 will choose the safe project if  $x_1(h) \geq \frac{25}{7}$  resulting in the investment profile  $(b, g, b)$ . Otherwise, the investment profile is  $(g, g, b)$ .

5. Bank 2 chooses the safe project independently of the choice of banks 1 and 3. Bank 3 chooses the safe project only if bank 1 chooses the safe project as well. This happens when  $x_2(h) \geq \frac{125}{28}$  and  $\frac{20}{7} \leq x_3(h) < \frac{25}{7}$ . Bank 1 chooses the safe project, even after the revision round, if  $x_1(h) \geq \max\{\frac{100}{63}, \frac{20}{7}\} = \frac{20}{7}$  resulting in the investment profile  $(b, b, b)$ . Otherwise, the investment profile is  $(g, b, g)$ .
6. Bank 3 chooses the safe project independently of the choice of banks 1 and 2. Bank 2 chooses the safe project only if bank 1 chooses the safe project as well. This happens when  $\frac{20}{7} \leq x_2(h) < \frac{25}{7}$  and  $x_3(h) \geq \frac{125}{28}$ . Bank 1 chooses the safe asset if  $x_1(h) \geq \max\{\frac{100}{63}, \frac{20}{7}\} = \frac{20}{7}$  resulting in the investment profile  $(b, b, b)$ . Otherwise, the investment profile is  $(g, g, b)$ .
7. Bank 2 chooses the gambling project independently of the investment choices of banks 1 and 3. Bank 3 chooses the safe project only if bank 1 chooses the safe project as well. This happens when  $x_2(h) < \frac{125}{63}$  and  $\frac{25}{7} \leq x_3(h) < \frac{125}{28}$ . Bank 1 chooses the safe project if  $x_1(h) \geq \max\{\frac{125}{63}, \frac{25}{7}\}$  resulting in the investment profile  $(b, g, b)$ . Otherwise, the investment profile is  $(g, g, g)$ .
8. Bank 3 chooses the gambling project independently of the investment choices of banks 1 and 2. Bank 2 chooses the safe project only if bank 1 chooses the safe project as well. This happens when  $\frac{25}{7} \leq x_2(h) < \frac{125}{28}$  and  $x_3(h) < \frac{20}{7}$ . Bank 1 chooses the safe project if  $x_1(h) \geq \max\{\frac{125}{63}, \frac{25}{7}\}$  resulting in the investment profile  $(b, b, g)$ . Otherwise, the investment profile is  $(g, g, g)$ .
9. Both banks 2 and 3 choose the safe project only if bank 1 chooses the safe project, while they both choose the gambling project if bank 1 chooses the gambling project. This happens when  $\frac{20}{7} \leq x_2(h) < \frac{125}{28}$  and  $\frac{20}{7} \leq x_3(h) < \frac{25}{7}$  and when  $\frac{20}{7} \leq x_2(h) < \frac{25}{7}$  and  $\frac{25}{7} \leq x_3(h) < \frac{125}{28}$ . Bank 1 chooses the safe project if  $x_1(h) \geq \max\{\frac{500}{427}, \frac{20}{7}\}$  resulting in the investment profile  $(b, b, b)$ . Otherwise, the investment profile is  $(g, g, g)$ .

The resulting investing profiles  $(s_1(h), s_2(h), s_3(h))$  are summarized as follows:

1.  $(b, b, b)$  if  $x_i(h) \geq \frac{20}{7}$  for all  $i = 1, 2, 3$
2.  $(b, b, g)$  if  $x_i(h) \geq \frac{20}{7}$  for  $i = 1, 2$  and  $x_3(h) < \frac{20}{7}$
3.  $(b, g, b)$  if  $x_i(h) \geq \frac{20}{7}$  for  $i = 1, 3$  and  $x_2(h) < \frac{20}{7}$
4.  $(g, b, b)$  if  $x_i(h) \geq \frac{20}{7}$  for  $i = 2, 3$  and  $x_1(h) < \frac{20}{7}$

5.  $(b, g, g)$  if  $x_i(h) < \frac{25}{7}$  for  $i = 2, 3$ , at least one  $i$  in  $\{2, 3\}$  with  $x_i(h) < \frac{20}{7}$  and  $x_1(h) \geq \frac{125}{28}$
6.  $(g, b, g)$  if  $x_i(h) < \frac{25}{7}$  for  $i = 1, 3$ , at least one  $i$  in  $\{1, 3\}$  with  $x_i(h) < \frac{20}{7}$  and  $x_2(h) \geq \frac{125}{28}$
7.  $(g, g, b)$  if  $x_i(h) < \frac{25}{7}$  for  $i = 1, 2$ , at least one  $i$  in  $\{1, 2\}$  with  $x_i(h) < \frac{20}{7}$  and  $x_3(h) \geq \frac{125}{28}$
8.  $(g, g, g)$  if  $x_i(h) < \frac{125}{28}$  for all  $i = 1, 2, 3$ , at least one  $i$  in  $\{1, 2, 3\}$  has  $x_i(h) < \frac{25}{7}$  and at least one  $j$  in  $\{1, 2, 3\}$ ,  $i \neq j$  has  $x_j(h) < \frac{20}{7}$ .

Consider now the transfer stage. Since bank 4 is disconnected in network  $K^*$ , it has no incentives to transfer its bank capital (no need to avoid counterparty risk). For the same reason, banks 1, 2 and 3 do not have incentives to transfer money to bank 4. Therefore, we will concentrate on histories  $h$  such that

$$x_1(h) + x_2(h) + x_3(h) = e_1 + e_2 + e_3 = \frac{65}{7}.$$

Furthermore, bank 1 will not make a transfer such that it will induce itself to switch from the safe project to the gambling one. Indeed, assume that bank 1 transfers an amount of bank capital such that one bank (either 2 or 3) chooses the safe project and bank 1 chooses the gambling project. Then  $x_1(h) < \frac{25}{7}$  and therefore we have

$$\eta^2 f(2) R x_1(h) + B < \frac{4}{5} \frac{4}{5} \frac{7}{4} R \frac{25}{7} + B = 4R + B = 5R. \quad (15)$$

This implies that bank 1 prefers autarky, where its expected payoff is  $5R$ , without joining the network. Similarly, assume that bank 1 transfers an amount of bank capital such that banks 2 and 3 choose the safe project and bank 1 chooses the gambling one. Then  $x_1(h) < \frac{20}{7}$  and therefore we have

$$\eta f(2) R x_1(h) + B < \frac{4}{5} \frac{7}{4} R \frac{20}{7} + B = 4R + B = 5R. \quad (16)$$

Again, bank 1 prefers autarky. For this reason, we consider histories such that bank 1 chooses the safe project. This allows us to consider only histories such that banks 2 and 3 transfer capital between each other but not to bank 1. Bank 1 instead can transfer capital to either bank 2 or bank 3 or both.

The decision by bank 3 to transfer capital depends on whether bank 2 chooses the gambling project without that transfer. Several cases are possible.

1. Assume that bank 3 chooses the gambling project before and after the transfer. Then  $x_2(h) < \frac{25}{7}$ ,  $x_3(h) < \frac{25}{7}$  with at least one of them smaller than  $\frac{20}{7}$  (which implies that  $x_2(h) + x_3(h) < \frac{45}{7}$ ). Then the minimum transfer has to be equal to  $\frac{25}{7} - x_2(h)$ , which means that  $x_2(h) + x_3(h) \geq \frac{25}{7}$ . Given that bank 3 chooses the gambling project after the transfer, we have  $x_2(h) + x_3(h) < \frac{40}{7}$ . Furthermore, for the transfer to be profitable for bank 3 it has to be that

$$\eta^2 f(2) R x_3(h) + B \leq \eta f(2) R (x_3(h) - t_3) + B,$$

which implies that bank 3 transfers at most  $\frac{1}{5}x_3(h)$  to bank 2.

2. Assume now that bank 3 chooses gambling before the transfer, but it switches to the safe project after the transfer. Then  $x_2(h) < \frac{25}{7}$ ,  $x_3(h) < \frac{25}{7}$  with at least one of them smaller than  $\frac{20}{7}$ . Then the minimum transfer has to be equal to  $\frac{20}{7} - x_2(h)$  which means that  $x_2(h) + x_3(h) \geq \frac{40}{7}$ . In order for the transfer to be profitable for bank 3 it has to be that

$$\eta^2 f(2) R x_3(h) + B \leq f(2) R (x_3(h) - t_3),$$

which implies that bank 3 transfers at most  $\frac{9}{25}x_3(h) - \frac{4}{7}$  to bank 2.

3. Assume now that bank 3 chooses the safe project before and after the transfer. Then  $x_2(h) < \frac{20}{7}$  and  $x_3(h) > \frac{25}{7}$ . The minimum transfer has to be equal to  $\frac{20}{7} - x_2(h)$  which means that  $x_2(h) + x_3(h) \geq \frac{40}{7}$ . In order for the transfer to be profitable for bank 3 it has to be that

$$\eta f(2) R x_3(h) \leq f(2) R (x_3(h) - t_3),$$

which implies that bank 3 transfers at most  $\frac{1}{5}x_3(h)$  to bank 2.

4. Finally, assume that bank 3 chooses the safe project before the transfer and the gambling project after the transfer. Then  $x_2(h) < \frac{20}{7}$ ,  $x_3(h) > \frac{25}{7}$ . The minimum transfer has to be equal to  $\frac{25}{7} - x_2(h)$  which means that  $x_2(h) + x_3(h) < \frac{40}{7}$ . In order for the transfer to be profitable for bank 3 it has to be that

$$\eta f(2) R x_3(h) \leq \eta f(2) R (x_3(h) - t_3) + B,$$

which implies  $t_3 \leq \frac{5}{7}$ . So we need  $\frac{25}{7} - x_2(h) \leq \frac{5}{7}$  or  $x_2(h) \geq \frac{20}{7}$ , a contradiction since bank 2 has to choose the gambling project before the transfer. So this case is not possible.

Since the decision of bank capital transfer by bank 2 is symmetric to the decision of bank 3, then the same conditions for each case apply. Finally, bank 1 must decide whether to make bank capital transfers. We have 3 cases:

1. Without transfers, bank 1 chooses the safe project and banks 2 and 3 both choose the gambling project since none of the conditions that will make transfers between banks 2 and 3 profitable will be satisfied (see the case where bank 3 chooses the gambling project before and after the transfer). In such a case, bank 1's expected payoff is  $\eta^2 f(2) e_1 = \frac{28}{5} R$ . Note that this is higher than the expected payoff in autarky  $5R$ .
2. If bank 1 chooses to transfer money so that one bank (say bank 2) chooses the safe project, the minimum transfer bank 1 has to make is equal to 1. Then, given that the conditions in item 1 of bank 3's strategies are satisfied, bank 3 would transfer  $\frac{1}{5} x_3(h)$  to bank 2 (namely,  $\frac{3}{7}$  dollars). In this way, bank 2 has the  $\frac{25}{7}$  dollars of bank capital needed to choose the safe project. Note that bank 3 chooses the gambling project before and after the transfer. In such a case, bank 1's expected payoff is again  $\eta f(2) (5 - 1) = \frac{28}{5} R$ .
3. Finally, if bank 1 chooses to transfer money to both banks 2 and 3 so that they choose the safe project, the minimum transfer bank 1 has to make is equal to  $\frac{10}{7}$ , equally split to each bank. In such a case, bank 1's expected payoff is  $f(2) (5 - \frac{10}{7}) = \frac{25}{4} R$  which is greater than the previous two. Then it is optimal for bank 1 to make both banks 2 and 3 choose the safe project.

Therefore, if the network structure is equal to the CFB  $K^*$ , the capital re-allocation after transfers would be equal to  $x^* = (\frac{25}{7}, \frac{20}{7}, \frac{20}{7}, \frac{5}{7})$ . This results in an investment profile equal to  $s^* = (b, b, b, g)$  and therefore (SPE) expected payoffs equal to  $v(K^*) = (\frac{25}{4} R, 5R, 5R, \frac{11}{7} R)$ . We proved that transfers are correct, and they are able to determine the CFB network. We are going to analyze if the CFB network  $K^*$  is stable, that is if there are no incentives to establish new links.

Take the same network structure  $K^*$  in which the connection between banks 3 and 4 has been added. Formally,  $K' = K^* \cup (34)$ , with  $K'_1 = \{2, 3\}$ ,  $K'_2 = \{1, 3\}$ ,  $K'_3 = \{1, 2, 4\}$  and  $K'_4 = \{3\}$ . Notice that there is not enough bank capital in the economy to make everybody choose the safe project in structure  $K'$ . Banks 1 and 2 need at least  $\frac{20}{7}$  dollars of capital (as in  $K^*$ ), bank 3 needs  $I^*(3, 0, \eta) = \frac{8}{3}$  and bank 4 needs  $I^*(1, 0, \eta) = \frac{10}{3}$ . However,

$$\frac{20}{7} + \frac{20}{7} + \frac{8}{3} + \frac{10}{3} = \frac{82}{7} > 10.$$

For the same reasons as in network  $K^*$ , bank 1 always chooses the safe project. Furthermore, if any bank  $i \neq 1$  wishes to do a transfer  $t_i$  to a neighbor so that it switches investment decisions from the gambling to the safe project, it has to be that

$$\eta^{g_i+1} f(k_i) R x_i + B \leq \eta^{g_i} f(k_i) R (x_i - t_i) + B, \quad (17)$$

if bank  $i$  chooses the gambling project with and without the transfer. Note that equation (17) is true if and only if

$$t_i \leq (1 - \eta) x_i = \frac{1}{5} x_i. \quad (18)$$

The same upper bound for the transfer  $t_i$  holds if bank  $i$  chooses the safe project with and without the transfer.

Following backward induction arguments as before, we arrive to the turn of bank 1 in the transfer stage. Bank 1 can choose four different transfer strategies.

1. No transfers. This results in bank 1 choosing the safe project and the other three banks choosing the gambling project. Bank 1 has an expected payoff equal to

$$\eta^2 f(2) R e_1 = \frac{4}{5} \frac{4}{5} \frac{7}{4} R 5 = \frac{28}{25} R.$$

2. To transfer money to make only bank 2 switch from the gambling to the safe project. Bank 1 knows that bank 3 will transfer at most one fifth of its endowment to bank 2 (see equation (18)), which is  $\frac{3}{7}$ . So bank 1 only needs to transfer 1 dollar of capital to get bank 2 to choose the safe project. Note that bank 2 will have  $\frac{25}{7} = I^*(2, 1, \eta)$  dollars of capital after transfers, given that  $\frac{15}{7} + \frac{3}{7} + 1 = \frac{25}{7}$ . Bank 1's expected payoff is then

$$\eta f(2) R (e_1 - 1) = \frac{4}{5} \frac{7}{4} R 4 = \frac{28}{25} R.$$

3. To transfer money to make only bank 3 switch from the gambling to the safe project. Bank 1 knows that both banks 2 and 4 will transfer at most one fifth of their endowment to bank 3 (see equation (18)), which is  $\frac{3}{7}$  from bank 2 and  $\frac{1}{7}$  from bank 4. So bank 1 needs to transfer  $\frac{61}{42}$  of bank capital to get bank 3 to choose the safe project. Note that bank 3 will have  $\frac{25}{6} = I^*(3, 2, \eta)$  dollars of capital after transfers, given that  $\frac{15}{7} + \frac{3}{7} + \frac{1}{7} + \frac{61}{42} = \frac{25}{6}$ . Since  $\frac{61}{42} > 1$ , bank 1 has a lower expected payoff than if transferring money only to bank 2:

$$\eta f(2) R \left( e_1 - \frac{61}{42} \right) < \frac{28}{25} R.$$

4. To transfer money to make both banks 2 and 3 switch from the gambling to the safe project. Bank 1 knows that bank 4 will transfer at most one fifth of its endowment to bank 3, which is equal to  $\frac{1}{7}$ . Hence, bank 1 needs to transfer  $\frac{5}{7}$  of capital to bank 2 and  $\frac{22}{21}$  of capital to bank 3, a total of  $\frac{37}{21}$ . Note that bank 2 will have a total of  $\frac{20}{7} = I^*(2, 0, \eta)$  dollars of capital after transfers while bank 3 will have  $\frac{10}{3} = I^*(3, 1, \eta)$  dollars of capital after transfers, given that  $\frac{15}{7} + \frac{1}{7} + \frac{22}{21} = \frac{10}{3}$ . Bank 1's expected payoff is then

$$f(2)R \left( e_1 - \frac{37}{21} \right) = \frac{7}{4}R \frac{68}{21} = \frac{17}{3}R > \frac{28}{5}R.$$

Therefore, bank 1 chooses to transfer capital to both banks 2 and 3 and bank 4 transfers capital to bank 3. The allocation of capital after transfers is equal to  $\hat{x} = \left( \frac{68}{21}, \frac{20}{7}, \frac{10}{3}, \frac{4}{7} \right)$ . Expected payoffs are  $v(\hat{K}) = \left( \frac{17}{3}R, 5R, 5R, \frac{59}{35}R \right)$ . Given that bank 4 strictly prefers  $\hat{K}$ , while bank 3 is indifferent between  $K^*$  and  $\hat{K}$ , the CFB is not pairwise stable with transfers. Intuitively, bank 4 has a strong incentive to create a link with bank 3, and bank 3 will not disagree with building the link. Bank 3 is ready to accept the risk coming from bank 4 since it knows the system of transfers will compensate it for the higher risk. In particular, bank 1 is the ultimate payer for the higher risk, while it is not involved in the decision concerning the creation of that link. With this example we have shown that banks do not have incentive to form the CFB network when transfers are allowed.

## 5 Conclusions

We present a model of financial network formation and characterize the set of optimal financial networks as core-periphery structures. The optimal networks get more and more connected as the counterparty risk becomes smaller and smaller. The decentralized financial networks are also core-periphery structures, although the size of the core may not coincide with the optimal one. However, the decentralized system resembles the social planner's solution when counterparty risk is negligible. On the contrary, when the counterparty risk is not so low an inefficient decentralized financial network arises. This network externality is not internalized allowing for bank capital transfers. Some clarifications about our assumptions are in order.

Bloch and Jackson [7] find that efficient networks are supported, although not uniquely, by pairwise stable equilibria of the *contingent transfer game*. In such a game agents make transfers to any other agent contingent on the network formation: Transfers are made to induce (because of positive externalities) or to deter (because of negative externalities) the creation of links. As shown in this paper, negative externalities do happen in our model.

Accordingly, also in our model efficient networks can be achieved by imposing a system of transfers contingent on the network formation. This implies that transfers should be made *before* the financial network is chosen. We believe that such a system of transfers is not feasible in reality since it would require a high level of coordination among banks.

We capture the benefits of obtaining coinsurance in the financial network with the function  $f(k_i)$ . This specification implicitly assumes that all banks have access to the same coinsurance independently of the connections of their counterparts. For example, consider three banks and two structures. While in the first structure banks 1 and 3 are disconnected but both connected to bank 2, in the second structure all banks are directly connected. In our setting, bank 2 has the same access to coinsurance in both networks. However, it could be that bank 2 coinsurance becomes weaker or stronger when banks 1 and 3 are directly connected.<sup>11</sup> In our model, allowing for such possibility implies that the function  $f(\cdot)$  should have more than one argument. However, under the assumption that  $f(\cdot)$  is increasing with respect to  $k_i$ , the results on the core-periphery structure still hold. Nevertheless, we consider this an interesting avenue for future research in network formation games.

We focus our attention on the contagion effects of direct links. Clearly, in the analysis there could be included also cascades or systemic effects, which would amplify the effects studied in this paper. Since we consider this an interesting and relevant topic for network formation, we leave it for future research. However, this paper points out that these systemic effects are not necessary in order to have inefficiencies in a decentralized financial network.

Finally, a crucial topic for future research concerns the implication of the network analysis for financial regulation. Our results suggest that decentralized networks form structures that resemble the CFB core-periphery structure but the decentralized network can be inefficiently underconnected. All else equal, a public guarantee in the markets where banks find liquidity coinsurance should improve the efficiency of the decentralized network. However, these kind of policies also deteriorate the moral hazard problem. Clearly, more work remains to be done to understand the appropriate instruments to address networks inefficiencies.

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<sup>11</sup>Castiglionesi and Wagner [10] provide a three-bank model where this issue is analyzed.

# Appendix A

## Microfoundations of the model.

First, we show under what conditions banks do not find it optimal to self-insure their liquidity uncertainty (i.e., to hold an amount of liquidity that precludes the coinsurance benefits provided by the network). Let's indicate with  $l$  the amount of liquidity held by bank  $i$  to face the liquidity shocks  $\omega_H$  and  $\omega_L$ , with  $\omega_H > \omega_L$ . We have the following

**Proposition 11** *Assume  $\omega_H \geq \bar{\omega}_H$ , then bank  $i$  does not find it optimal to self-insure.*

**Proof.** In order to establish the optimal liquidity holding, we distinguish three cases for bank  $i$ 's investment in liquidity:

- Invest  $l < \omega_L$ . In this case bank  $i$  cannot deal with the liquidity shock, and the project is foregone. This means that the bank will have an expected payoff of  $l$  for each dollar invested, which in turn implies that bank  $i$  finds it optimal to maximize liquidity holding choosing  $l = 1$ . But this contradicts with  $l < \omega_L < 1$ , so it is not optimal to invest  $l < \omega_L$ .
- Invest  $\omega_L \leq l < \omega_H$ . In this case there is the possibility of coinsurance if bank  $i$  finds a counterpart among its neighbors. The probability to find the counterpart is  $\varphi_i(k_i)$  that for simplicity we indicate with  $\varphi$ . In this case if bank  $i$  has a high shock it can withdraw from a bank (or the banks) with a low shock and then the funds move in the opposite direction when the project matures. That is, the amount exchanged in  $t = 3$  between bank  $i$  and its counterpart(s) is  $l - \omega_L = \omega_H - l$ , which implies that the optimal amount of liquidity to hold is  $l = (\omega_L + \omega_H)/2 \equiv \gamma$ . In this case the expected return in bank  $i$  for each dollar invested is

$$D_{i(l=\gamma)} = \frac{1-\varphi}{2}(\gamma) + \frac{1-\varphi}{2}[(1-\gamma)\bar{R} + \gamma - \omega_L] + \frac{\varphi}{2}[(1-\gamma)\bar{R} + \gamma - \omega_L] + \frac{\varphi}{2}[(1-\gamma)\bar{R} - (\omega_H - \gamma)]$$

this expression captures the fact that when the coinsurance is not possible because there was no counterpart with a different shock, then bank  $i$  either earns  $\gamma$  in case the high shock hits the bank's project, or  $(1-\gamma)\bar{R} + \gamma - \omega_L$  if the low shock hits the bank's project. When coinsurance instead is possible then bank  $i$  either gets  $(1-\gamma)\bar{R} + \gamma - \omega_L$  when the low shock appears, or  $(1-\gamma)\bar{R} - (\omega_H - \gamma)$  when the high shock hits. The former expression can be rewritten as

$$D_{i(l=\gamma)} = \frac{1+\varphi}{2}[(1-\gamma)\bar{R}] + \frac{1-\varphi}{2}\omega_H.$$

- Invest  $\omega_H \leq l$ . In this case bank  $i$  finds optimal to fix  $l = \omega_H$  since the return of holding liquidity (i.e., 1) is dominated by the return of the project  $\bar{R} > 1$ . When bank  $i$  sets liquidity at this high level there is no need of finding coinsurance. The expected payoff in bank  $i$  for each dollar invested is

$$D_{i(l=\omega_H)} = (1 - \omega_H)\bar{R} + \frac{\omega_H - \omega_L}{2},$$

since in this case bank  $i$  gains the return on the project plus the liquidity rolled over in case the shock is low (which happens with probability 1/2).

We need  $D_{i(l=\gamma)} > D_{i(l=\omega_H)}$ . Assuming that  $D_{i(l=\gamma)}$  is increasing in  $\varphi$  (see next proposition for a sufficient condition), and indicating with  $R$  (as in the text) the expected payoff when bank  $i$  is in autarky, that is when  $\varphi = 0$ , we need that  $R > D_{i(l=\omega_H)}$ . Given that

$$R \equiv D_{i(l=\gamma)}^{\varphi=0} = \frac{1}{2}[(1 - \gamma)\bar{R} + \omega_H], \quad (19)$$

the condition  $R > D_{i(l=\omega_H)}$  implies

$$\bar{R}(3\omega_H - 2 - \omega_L) > -2\omega_L.$$

The previous inequality is always satisfied if the expression in brackets is non-negative, that is when

$$\omega_H \geq \frac{2 + \omega_L}{3} \equiv \bar{\omega}_H. \quad (20)$$

■

The intuition is straightforward. When the bad liquidity shock is sufficiently high it is too costly to self-insure and bank  $i$  would not do that even in autarky. We also check our claim in the text that  $R < \bar{R}$ . Indeed

$$\frac{1}{2}[(1 - \gamma)\bar{R} + \omega_H] < \bar{R}$$

can be written as

$$(1 - \gamma)\bar{R} + \omega_H < \bar{R} + \bar{R},$$

which is trivially satisfied since  $\omega_H < 1 < \bar{R}$ , and  $(1 - \gamma)\bar{R} < \bar{R}$ . Clearly the liquidity shocks do not allow the banks to reap the full return of the illiquid project  $\bar{R}$ . However, the coinsurance mechanism provided by the network can increase the expected payoff. We have the following

**Proposition 12** *Assume  $\bar{R} > R^*$ , then the expected payoff  $D_{i(l=\gamma)}$  is increasing in  $\varphi$ .*

**Proof.** We can rewrite the expected payoff  $D_{i(l=\gamma)}$  as follows

$$D_{i(l=\gamma)} = \frac{1}{2}[(1 - \gamma)\bar{R} + \omega_H] - \frac{\varphi}{2}[\omega_H - (1 - \gamma)\bar{R}]$$

that is increasing in  $\varphi$  as long as  $(1 - \gamma)\bar{R} > \omega_H$  that implies

$$\bar{R} > \frac{2\omega_H}{2 - \omega_H - \omega_L} \equiv R^*. \quad (21)$$

■

Also in this case the intuition is simple. An expected return of the illiquid project sufficiently high makes the coinsurance mechanism valuable. We can write now the expected payoff  $D_{i(l=\gamma)}$  as a function of  $R$  and  $\varphi$ . Given condition (19) we have

$$D_{i(l=\gamma)} = R + \frac{\varphi}{2}[2R - 2\omega_H] = R + \varphi[R - \omega_H]$$

which is the same expression as in (1) beside the constant  $\omega_H$ . Notice that the inequality  $R > \omega_H$  is true under the assumption in Proposition 12. Indeed,  $R > \omega_H$  implies that  $(1 - \gamma)\bar{R} > \omega_H$ , which holds whenever condition (21) holds.

Finally, we analyze the transmission mechanism of bankruptcy. A bank is bankrupt when it cannot serve its depositors the amount of money promised in the deposit contracts. We assume contingent deposit contract on the occurrence of the realization of the liquidity shocks and the possibility of finding coinsurance. So a bank does not default because of the high liquidity shock or because it does not find coinsurance. A bank can then default either when it invests in the gambling project or when the counterpart has invested in the gambling project. Indeed, the links established by banks represent credit lines that allow them to withdraw money from the counterpart whenever a bank has the high liquidity shock. In this case, if a bank is creditor of a bankrupt gambling bank it is negatively affected. Consider bank  $i$  and the contingent deposit contract it offers to its depositors when it is optimal to invest  $l = \gamma$  (so there is the possibility of coinsurance). We have four possible payoffs:

- $\gamma$  is the amount of money promised when a counterpart with a different shock is not among the neighbors and its own liquidity shock is high (the return of the project is lost and only liquidity can be given to depositors);
- $(1 - \gamma)\bar{R} + \gamma - \omega_L$  is the amount of money promised when a counterpart with a different shock is not among its neighbors and its own liquidity shock is low;
- $(1 - \gamma)\bar{R} - (\omega_H - \gamma)$  is the amount of money promised when a counterpart with a different shock is found and its own liquidity shock is high;

- $(1 - \gamma)\bar{R} + \gamma - \omega_L$  is the amount of money promised when a counterpart with a different shock is found and its own liquidity shock is low.

Clearly the only case in which bank  $i$  cannot serve the promised amount of money to its depositors (because of the failure of its neighbor) is the fourth case. Bank  $i$  has a low liquidity shock and it has established a link with a counterpart with a high liquidity shock. The latter will withdraw the amount  $\gamma - \omega_L$  with the intention to give it back when its illiquid project matures. Suppose, however that the counterpart is a gambling bank that has zero return from its project (that is, it defaults). Taking into account also the payoff from its bank capital, bank  $i$  has  $(1 + x_i - \gamma)\bar{R}$  dollars to serve its depositors. Accordingly, bank  $i$  can avoid bankruptcy only if its bank capital is high enough to pay the amount of money promised to depositors. That is, bank  $i$  is bankrupt if

$$x_i \bar{R} < \gamma - \omega_L \implies \bar{R} < \frac{\omega_H - \omega_L}{2x_i}.$$

Recalling condition (21), we need

$$\frac{2\omega_H}{2 - \omega_H - \omega_L} < \frac{\omega_H - \omega_L}{2x_i}$$

which implies

$$x_i < \frac{(2 - \omega_H - \omega_L)(\omega_H - \omega_L)}{4\omega_H} \equiv \bar{x}. \quad (22)$$

Notice that all the expressions in brackets are positive so  $\bar{x} > 0$ . Then, whenever  $0 < x_i < \bar{x}$ , the bankruptcy is transmitted to bank  $i$ . Given that the probability of default of the counterpart is equal to  $(1 - \xi)$ , bank  $i$  defaults with probability  $\frac{(1-\xi)}{4}$  which we indicated as  $1 - \eta$  in the text. It immediately follows that  $\eta > \xi$ . Clearly, the higher the number of gambling neighbors then the higher the probability is for bank  $i$  to end up insuring with a risky counterpart. Then the probability of bank  $i$  going bankrupt is increasing in  $g_i$ . We capture this effect assuming that the probability of bank  $i$  to default is  $1 - \eta^{g_i}$  (so probability of surviving is  $\eta^{g_i}$ ). This assumption allows for a well defined probability (between 0 and 1).

## Appendix B

**Proof of Proposition 1.** The existence of a CFB can be proved in two steps. First, fixing each network  $K$  and each vector of investments  $s$  such that  $p_i(K, s)f(k_i)R \geq 1$  (constraint

9) and maximizing the objective function subject to

$$x_i \geq 0 \text{ for all } i \in N \quad (23)$$

$$\sum_{i \in N} x_i = \sum_{i \in N} e_i \equiv E \quad (24)$$

$$x_i \geq I^*(k_i, g_i, \xi, \eta) \text{ if } s_i = b \quad (25)$$

$$x_i < I^*(k_i, g_i, \xi, \eta) \text{ if } s_i = g. \quad (26)$$

Second, once the maximum total expected payoff is obtained given  $K$  and  $s$ , it suffices to choose the combination of  $K$  and  $s$  that has the highest expected total payoff. Notice that the second step does not create problem of existence as we are choosing the highest number on a discrete set of numbers. However, in the first step the problematic restriction is (25). Let us modify the planner problem by rewriting such constraint as

$$x_i \leq I^*(k_i, g_i, \xi, \eta) \text{ if } s_i = g.$$

In the modified problem, a maximum always exists. Furthermore, the solution of the modified problem is also a maximum of the social planner's problem. If this were not true, then there is an economy for which the solution of the modified problem, a triple  $(K^*, x^*, s^*)$ , is such that  $x_i^* = I^*(k_i, g_i, \xi, \eta)$  for at least one bank  $i$  with  $s_i = g$ . But if  $x_i^* = I^*(k_i, g_i, \xi, \eta)$  for at least one bank  $i$  then the triple  $(K^*, x^*, s')$ , where  $s'$  and  $s^*$  differ only on the choice of investment by bank  $i$ , is also feasible and yields a higher expected payoff to at least bank  $i$  and its neighbors (and no bank gets lower expected payoff) than in  $s^*$ . Therefore,  $(K^*, x^*, s^*)$  could not have been a maximum of the modified problem, a contradiction.

Concerning the second statement, we check the incentive and participation constraints. First note that if  $\sum_{i \in N} e_i = E \geq \frac{n-1}{\rho-1} \frac{\rho B}{(1-\xi)R}$  there is enough bank capital to satisfy the incentive constraints. This is so given that

$$\frac{n-1}{\rho-1} \frac{\rho B}{(1-\xi)R} \geq \frac{nB}{(1-\xi)\rho R},$$

for  $1 < \rho \leq 2 \leq n$ . In terms of the participation constraints the higher the initial endowment of a bank the harder it is for the planner to satisfy the participation constraint. The most difficult case is when one bank is endowed with all the bank capital in the economy  $E$  and the remaining banks have 0 bank capital.

If  $n \geq 2 \geq \rho$  then  $E \geq \frac{nB}{(1-\xi)\rho R}$  implies that  $E \geq \frac{B}{(1-\xi)R}$  and the 'all-endowed' bank chooses the safe project in autarky. The rest of the banks choose the gambling project in autarky. Therefore, the planner needs to give  $\max\{\frac{B}{(1-\xi)R}, \frac{E}{\rho}\}$  dollars of bank capital

to the highly endowed bank and  $\max\{\frac{B}{(1-\xi)R}, \frac{B}{\rho R}\}$  to each of the other banks. Note that  $\frac{E}{\rho} \geq \frac{B}{(1-\xi)R} \geq \frac{B}{\rho R}$ , for  $1 < \rho \leq 2 \leq n$ . Therefore, the planner needs  $E$  to be at least equal to  $\frac{E}{\rho} + (n-1) \frac{B}{(1-\xi)R}$ , or, rearranging  $E \geq \frac{(n-1)}{(\rho-1)} \frac{\rho B}{(1-\xi)R}$ . If the banks had the same bank capital endowment, we leave to the reader to check that the condition for INE is sufficient to induce investors of all banks to participate. ■

**Proof of Proposition 2.** The first statement is proved by contradiction. Assume that  $(K^*, x^*, s^*)$  is a CFB for  $(N, e)$  such that there is a bank  $i$  with  $s_i^* = g$  and  $x_i^* > x_i^A(K^*, s^*)$ , and there is another bank  $j$  with  $k_j^* \geq k_i^*$ ,  $s_j^* = b$  and  $g_j \geq g_i$  (the proof of the case for  $s_j^* = g$  and  $g_j < g_i$  is equivalent and therefore omitted). Take  $(K^*, \hat{x}, s^*)$  a new allocation for the same economy  $(N, e)$ , where the network and investment strategies are the same. The allocation of capital differs in bank  $j$  receiving all the “extra” (i.e. above payoff equivalent to autarky) capital endowment of bank  $i$ . Formally,  $\hat{x}_i = x_i^A(K^*, s^*)$ ,  $\hat{x}_j = x_j^* + x_i^* - x_i^A(K^*, s^*)$ , and  $\hat{x}_r = x_r^*$  for all  $r \neq i, j$ . We show that  $(K^*, \hat{x}, s^*)$  is an INE, satisfies the participation constraints and yields a higher expected total money generated in the economy. Therefore, the initial allocation  $(K^*, x^*, s^*)$  cannot be a solution to the planners problem and statement 1 follows.

- Note that if  $(K^*, x^*, s^*)$  is a CFB then it has to be an INE. We check now that  $(K^*, \hat{x}, s^*)$  is also an INE. Consider bank  $i$ . Since  $(K^*, x^*, s^*)$  is an INE and  $s_i^* = g$  it has to be that  $x_i^* < I^*(k_i^*, g_i, \xi, \eta)$ . By assumption,  $\hat{x}_i = x_i^A(K^*, s^*) < x_i^*$ , and therefore  $\hat{x}_i < I^*(k_i^*, g_i, \xi, \eta)$ . Consider now bank  $j$ . As  $j$  is choosing the safe project in  $(K^*, x^*, s^*)$ , we know that  $x_j^* \geq I^*(k_j^*, g_j, \xi, \eta)$ . By definition of  $\hat{x}$ ,  $\hat{x}_j > x_j^* \geq I^*(k_j^*, g_j, \xi, \eta)$ . Therefore the allocation  $(K^*, \hat{x}, s^*)$  is an INE.
- Note that, since  $(K^*, x^*, s^*)$  is a CFB, the participation constraint for investors is satisfied. By definition of  $\hat{x}$  the participation constraint is thus satisfied for any bank different than  $i$  and  $j$ . Furthermore, given that  $\hat{x}_i = x_i^A(K^*, s^*)$ ,  $i$  trivially satisfies the participation constraint in  $(K^*, \hat{x}, s^*)$ . Finally, since  $\hat{x}_j > x_j^*$  we know that the participation constraint has to be satisfied for bank  $j$  in  $(K^*, \hat{x}, s^*)$ . This means that the participation constraint is satisfied for all banks in  $(K^*, \hat{x}, s^*)$ .
- Depositors expected payoffs in  $(K^*, \hat{x}, s^*)$  are the same as in  $(K^*, x^*, s^*)$ . Therefore, if  $(K^*, x^*, s^*)$  was a CFB it had to satisfy the participation constraint for depositors. Thus, the participation constraint for depositors is trivially satisfied in  $(K^*, \hat{x}, s^*)$ .

- Finally,

$$\begin{aligned} & \sum_{r \in N} [m_r(K^*, \hat{x}_r, s^*) + M_r(K^*, s^*)] - \sum_{r \in N} [m_r(K^*, x^*, s^*) + M_r(K^*, s^*)] \\ &= R(x_i^* - x_i^A(K^*, s^*)) [\eta^{g_j} f(k_j^*) - \xi \eta^{g_i} f(k_i^*)] \geq 0, \end{aligned} \quad (27)$$

given that  $k_j^* \geq k_i^*$ , the function  $f(k)$  is increasing in  $k$ ,  $g_j \leq g_i$  and  $\xi \leq \eta \leq 1$ . By equation (27), the allocation  $(K^*, \hat{x}, s^*)$  yields a higher expected payoff in the economy. Therefore  $(K^*, x^*, s^*)$  was not a CFB.

Also the second statement is proved by contradiction. Assume that  $(K^*, x^*, s^*)$  is a CFB for  $(N, e)$  but there is a bank  $i$  such that  $s_i^* = b$  and  $x_i^* > \max \{x_i^A(K^*, x^*), I^*(k_i^*, g_i, \xi, \eta)\}$ , and there is another bank  $j$  with  $k_j^* \geq k_i^*$  such that  $s_j^* = b$  and  $g_j < g_i$  (the proof of the case for  $s_j^* = g$  and  $g_j < g_i - 1$  is equivalent and therefore omitted). Take  $(K^*, \hat{x}, s^*)$  a new allocation for the same economy  $(N, e)$ , where the network and investment strategies are the same. The allocation of capital differs in bank  $j$  receiving all “extra” capital endowment of bank  $i$ . Formally,  $\hat{x}_i = \max \{x_i^A(K^*, x^*), I^*(k_i^*, g_i, \xi, \eta)\}$ ,  $\hat{x}_j = x_j^* + x_i^* - \hat{x}_i$ , and  $\hat{x}_r = x_r^*$  for all  $r \neq i, j$ . We show now that  $(\hat{x}, K^*, s^*)$  is an INE, it satisfies the participation constraint for any bank and it yields a higher expected payoff in the economy. Therefore, the initial allocation  $(K^*, x^*, s^*)$  cannot be a solution to the planner’s problem and statement 2 follows.

- As before, since  $(K^*, x^*, s^*)$  is a CFB then it has to be an INE. We show now that  $(K^*, \hat{x}, s^*)$  is an INE. Note that, by definition of  $\hat{x}$  (i)  $\hat{x}_i \geq I^*(k_i^*, g_i, \xi, \eta)$  and (ii)  $\hat{x}_j > x_j^*$ . Given that  $(K^*, x^*, s^*)$  is a INE,  $x_j^* \geq I^*(k_j^*, g_j, \xi, \eta)$  and therefore  $\hat{x}_j > I^*(k_j^*, g_j, \xi, \eta)$ . This means that the allocation  $(K^*, \hat{x}, s^*)$  is an INE.
- As before, since  $(K^*, x^*, s^*)$  is a CFB, the participation constraint for investors is satisfied. By definition of  $\hat{x}$  the participation constraint is trivially satisfied for any bank different than  $i$  and  $j$ . Furthermore, given that  $\hat{x}_i = \max \{x_i^A(K^*, x^*), I^*(k_i^*, g_i, \xi, \eta)\}$ ,  $i$  trivially satisfies the participation constraint in  $(K^*, \hat{x}, s^*)$ . Finally, since  $\hat{x}_j > x_j^*$  we know that the participation constraint has to be satisfied for bank  $j$  in  $(K^*, \hat{x}, s^*)$ . This means that the participating constraint is satisfied for all banks in  $(K^*, \hat{x}, s^*)$
- The participation constraint for depositors is trivially satisfied, since depositors obtain the same in  $(K^*, x^*, s^*)$  than in  $(K^*, \hat{x}, s^*)$ .

- Finally,

$$\begin{aligned} & \sum_{r \in N} [m_r(K^*, \hat{x}_r, s^*) + M_r(K^*, s^*)] - \sum_{r \in N} [m_r(K^*, x^*, s^*) + M_r(K^*, s^*)] \\ &= R(x_i^* - \hat{x}_i) [\eta^{g_j} f(k_j^*) - \eta^{g_i} f(k_i^*)] \geq 0, \end{aligned} \quad (28)$$

given that  $k_j^* \geq k_i^*$ , the function  $f(k)$  is increasing in  $k$  and  $g_j \leq g_i$ . By equation (28), the allocation  $(K^*, \hat{x}, s^*)$  yields a higher expected payoff in the economy and therefore  $(K^*, x^*, s^*)$  was not a CFB. ■

**Proof of Proposition 3.** The proof is made by contradiction. Assume that  $(K^*, x^*, s^*)$  is a CFB for  $(N, e)$  but there are two unconnected banks  $i$  and  $j$  such that  $s_i^* = s_j^* = b$ . Take a new allocation  $(\hat{K}, x^*, s^*)$  for the same economy  $(N, e)$ , where the allocation of bank capital and strategies are the same but the network structure only adds the link between banks  $i$  and  $j$ . Formally,  $\hat{K}_i = K_i^* \cup \{j\}$ ,  $\hat{K}_j = K_j^* \cup \{i\}$ , and  $\hat{K}_r = K_r^*$  for all  $r \neq i, j$ . We show now that the allocation  $(x^*, \hat{K}, s^*)$  is an INE, it satisfies the participation constraints for any bank and it yields a higher expected total payoff. Therefore, the initial allocation  $(K^*, x^*, s^*)$  cannot be a solution to the planner's problem and Proposition 3 follows.

Note first that, by definition of  $\hat{K}$ ,  $p_r(\hat{K}, s^*) = p_r(K^*, s^*)$ , for all  $r \in N$ , and

$$\hat{k}_r = \begin{cases} k_r^* + 1, & \text{if } r = i \text{ or } r = j \\ k_r^*, & \text{otherwise.} \end{cases}$$

- We first show that  $(\hat{K}, x^*, s^*)$  is an INE. Note that  $(K^*, x^*, s^*)$  is an INE given that it is a CFB. This means, since both  $i$  and  $j$  are choosing the safe project, that  $x_i^* \geq I^*(k_i^*, g_i, \xi, \eta)$  and  $x_j^* \geq I^*(k_j^*, g_j, \xi, \eta)$ . By definition of the functions  $I^*$ , it is true then that  $x_i^* \geq I^*(k_i^* + 1, g_i, \xi, \eta)$  and  $x_j^* \geq I^*(k_j^* + 1, g_j, \xi, \eta)$ . Therefore, the allocation  $(\hat{K}, x^*, s^*)$  is an INE.
- We show now that  $(\hat{K}, x^*, s^*)$  satisfies the investors participation constraints in any bank. Given that  $(K^*, x^*, s^*)$  satisfies the participation constraints for any bank, it is true that  $x_i^* \geq x_i^A(K^*, x^*)$  and  $x_j^* \geq x_j^A(K^*, x^*)$ . Recall that: (i)  $x_i^A(K^*, x^*) = \frac{m_i^A}{\eta^{g_i} f(k_i^*) R}$  and  $x_j^A(K^*, x^*) = \frac{m_j^A}{\eta^{g_j} f(k_j^*) R}$ ; (ii)  $x_i^A(\hat{K}, x^*) = \frac{m_i^A}{\eta^{g_i} f(k_i^* + 1) R}$  and  $x_j^A(\hat{K}, x^*) = \frac{m_j^A}{\eta^{g_j} f(k_j^* + 1) R}$ . Given that  $f(k)$  is an increasing function in  $k$ , it is true that  $x_i^A(K^*, x^*) > x_i^A(\hat{K}, x^*)$  and  $x_j^A(K^*, x^*) > x_j^A(\hat{K}, x^*)$ . Therefore  $x_i^* \geq x_i^A(\hat{K}, x^*)$  and  $x_j^* \geq x_j^A(\hat{K}, x^*)$ . This means that the allocation  $(\hat{K}, x^*, s^*)$  satisfies the investors participation constraints in any bank.
- In order to see that the participation constraints for depositors are satisfied in  $(\hat{K}, x^*, s^*)$  the following statements are sufficient. First, note that, by assumption,  $(K^*, x^*, s^*)$  satisfies the participation constraints for depositors, given that it is a CFB. This means that  $\eta^{g_r} f(k_r^*) R \geq 1$  for all banks  $r$ . Since the function  $f(k)$  is increasing in  $k$ , it has to be that  $\eta^{g_i} f(k_i^* + 1) R \geq 1$  and  $\eta^{g_j} f(k_j^* + 1) R \geq 1$ .

- Finally,

$$\begin{aligned} & \sum_{r \in N} \left[ m_r \left( \hat{K}, x_r^*, s^* \right) + M_r \left( \hat{K}, s^* \right) \right] - \sum_{r \in N} \left[ m_r \left( K^*, x_r^*, s^* \right) + M_r \left( K^*, s^* \right) \right] \\ &= \eta^{g_i} R \left( x_i^* + 1 \right) \left[ f \left( k_i^* + 1 \right) - f \left( k_i^* \right) \right] + \eta^{g_j} R \left( x_j^* + 1 \right) \left[ f \left( k_j^* + 1 \right) - f \left( k_j^* \right) \right] > 0 \end{aligned} \quad (29)$$

since the function  $f(k)$  is increasing in  $k$ . By equation (29), the allocation  $(\hat{K}, x^*, s^*)$  yields higher expected payoff in the economy and therefore  $(K^*, x^*, s^*)$  was not a CFB. ■

**Proof of Proposition 4.** We prove that if an allocation is an INE and it satisfies the investors participation constraints in any bank, then it has to be that any bank choosing the safe project in autarky chooses the safe project in this allocation as well. Take any bank  $i$  such that  $\max\{Re_i, \xi Re_i + B\} = Re_i$ , or such that it chooses the safe project in autarky. Assume by contradiction that there exists an allocation  $(K, x, s)$  that is an INE, it satisfies the participation constraints for any bank and  $s_i = g$ . This implies then:

1.  $x_i < I^*(k_i, g_i, \xi, \eta)$  since  $(K, x, s)$  is an INE, and
2.  $\xi \eta^{g_i} f(k_i) R x_i + B \geq Re_i$  since  $(K, x, s)$  satisfies the participation constraint.

Given that bank  $i$  chooses the safe project in autarky,  $Re_i \geq \frac{B}{(1-\xi)}$ . The last condition together with the participation constraint for bank  $i$  (item 2 above) implies that

$$\xi \eta^{g_i} f(k_i) R x_i + B \geq \frac{B}{(1-\xi)},$$

and rearranging terms, we have

$$x_i \geq \frac{B}{(1-\xi) \eta^{g_i} f(k_i) R} = I^*(k_i, g_i, \xi, \eta)$$

a contradiction with  $(K, x, s)$  being an INE (item 1 above). Therefore, any bank  $i$  choosing the safe project in autarky chooses the safe asset in any INE satisfying the participation constraint (for at least that bank  $i$ ). ■

**Proof of Proposition 5.** We prove all statements by contradiction.

*Proof of Statement 1.*

Assume that  $(K^*, x^*, s^*)$  is a CFB allocation where there are two banks  $i$  and  $j$  not directly connected and such that  $k_i < \underline{k}(\eta)$  and  $k_j < \underline{k}(\eta)$ . Take a new allocation  $(\hat{K}, x^*, s^*)$  for the same economy  $(N, e)$ , where the allocation of capital and strategies are the same

but the network structure adds the link between banks  $i$  and  $j$ . Formally,  $\hat{K}_i = K_i^* \cup \{j\}$ ,  $\hat{K}_j = K_j^* \cup \{i\}$ , and  $\hat{K}_r = K_r^*$  for all  $r \neq i, j$ .

Note that if  $(K^*, x^*, s^*)$  is a CFB then it has to be an INE and to satisfy the participation constraints for depositors and investors in any bank. If at least one of them, for example  $i$ , is choosing safe (the equivalent applies for  $j$ ), then  $x_i^* \geq I^*(k_i^*, g_i, \xi, \eta)$ . The other bank, say  $j$ , is choosing the gambling project since we know from Proposition 3 that both banks cannot invest in the safe project otherwise  $(K^*, x^*, s^*)$  would not be a CFB allocation.

Note that  $I^*(k_i^*, g_i, \xi, \eta) \geq I^*(k_i^* + 1, g_i + 1, \xi, \eta)$  if and only if  $\eta f(k_i^* + 1) \geq f(k_i^*)$ , which is true for  $\eta \geq \frac{f(\underline{k}(\eta) - 1)}{f(\underline{k}(\eta))}$  and  $k_i \leq \underline{k}(\eta) - 1$ . We also have that the ratio  $\frac{f(k_i)}{f(k_i + 1)}$  is increasing in  $k_i$  and

$$\frac{f(\underline{k}(\eta) - 1)}{f(\underline{k}(\eta))} \geq \frac{f(k_i)}{f(k_i + 1)} \text{ for } k_i \leq \underline{k}(\eta) - 1.$$

Therefore, the allocation  $(\hat{K}, x^*, s^*)$  is an INE as far as  $x_j < I^*(k_i + 1, g_i, \xi, \eta)$ . If it were not, there is an allocation  $(\hat{K}, x^*, \hat{s})$  in which bank  $j$  chooses the safe project instead of the gambling one. This would be an INE for bank  $i$  since  $I^*(k_i^* + 1, g_i + 1, \xi, \eta) > I^*(k_i^* + 1, g_i, \xi, \eta)$ .

We show first that the participation constraint is satisfied by  $(\hat{K}, x^*, s^*)$  for all banks. Later, we show that  $(\hat{K}, x^*, s^*)$  yields a higher total expected payoff than  $(K^*, x^*, s^*)$ . Therefore, the initial allocation  $(K^*, x^*, s^*)$  cannot be a solution to the planner's problem and statement 1 follows. We separate cases depending whether bank  $i$  chooses the safe or the gambling project in the investment profile  $s^*$ .

### 1. Participation constraints:

- Assume first that bank  $i$  chooses the safe project in  $(K^*, x^*, s^*)$ . If  $\eta \geq \frac{f(\underline{k}(\eta) - 1)}{f(\underline{k}(\eta))}$  then  $\eta f(k_i^* + 1) > f(k_i^*)$  for  $k_i < \underline{k}(\eta)$ , which in turn implies that

$$\eta^{g_i+1} f(k_i^* + 1) R x_i^* > \eta^{g_i} f(k_i^*) R x_i^*,$$

and

$$\eta^{g_i+1} f(k_i^* + 1) R > \eta^{g_i} f(k_i^*) R.$$

Given that  $(K^*, x^*, s^*)$  satisfies the participation constraints for any bank, it has to be that  $\eta^{g_i} f(k_i^*) R x_i^* \geq m_i^A$  and  $\eta^{g_i} f(k_i^*) R \geq 1$ , and therefore,  $(\hat{K}, x^*, s^*)$  satisfies the participation constraints for bank  $i$ .

- If bank  $i$  chooses the gambling project in  $(K^*, x^*, s^*)$ , the argument is equivalent. Note that if  $\eta \geq \frac{f(\underline{k}(\eta)-1)}{f(\underline{k}(\eta))}$  then  $\eta f(k_i^* + 1) > f(k_i^*)$ , which in turn implies that

$$\xi \eta^{g_i+1} f(k_i^* + 1) R x_i^* + B \geq \xi \eta^{g_i} f(k_i^*) R x_i^* + B,$$

and

$$\xi \eta^{g_i+1} f(k_i^* + 1) R \geq \xi \eta^{g_i} f(k_i^*) R.$$

Given that  $(K^*, x^*, s^*)$  satisfies the participation constraints for any bank, it has to be that  $\xi \eta^{g_i} f(k_i^*) R x_i^* + B \geq m_i^A$  and  $\xi \eta^{g_i} f(k_i^*) R \geq 1$ , and therefore,  $(\hat{K}, x^*, s^*)$  satisfies the participation constraint for bank  $i$ .

- Checking that the participation constraint for bank  $j$ , which chooses the gambling project in  $(K^*, x^*, s^*)$ , is satisfied in  $(\hat{K}, x^*, s^*)$  works the same way as in the previous case (when  $i$  chooses the gambling project) and it is therefore omitted.

## 2. Higher total expected payoff:

- If bank  $i$  chooses the safe project in  $(K^*, x^*, s^*)$  we have:

$$\begin{aligned} & \sum_{r \in N} \left[ m_r(\hat{K}, x_i^*, s^*) + M_r(\hat{K}, s^*) \right] - [m_r(K^*, x_i^*, s^*) - M_r(K^*, s^*)] = \\ & = \eta^{g_i} R (x_i^* + 1) [\eta f(k_i^* + 1) - f(k_i^*)] + \xi \eta^{g_j} R (x_j^* + 1) [f(k_j^* + 1) - f(k_j^*)], \quad (30) \end{aligned}$$

- If banks  $i$  and  $j$  both choose the gambling project in  $(K^*, x^*, s^*)$  we have:

$$\begin{aligned} & \sum_{r \in N} \left[ m_r(\hat{K}, x_i^*, s^*) + M_r(\hat{K}, s^*) \right] - [m_r(K^*, x_i^*, s^*) + M_r(K^*, s^*)] = \\ & = \xi \eta^{g_i} R (x_i^* + 1) [\eta f(k_i^* + 1) - f(k_i^*)] + \xi \eta^{g_j} R (x_j^* + 1) [\eta f(k_j^* + 1) - f(k_j^*)]. \quad (31) \end{aligned}$$

Expressions (30) and (31) are greater than zero since  $\eta \geq \frac{f(k_i^*)}{f(k_i^* + 1)}$  and  $\eta \geq \frac{f(k_j^*)}{f(k_j^* + 1)}$  for  $k_i^* < \underline{k}(\eta)$  and  $k_j^* < \underline{k}(\eta)$ . Therefore, the allocation  $(\hat{K}, x^*, s^*)$  yields a higher total expected payoff and therefore  $(K^*, x^*, s^*)$  is not a CFB.

Furthermore, equation (30) implies that if  $s_i^* = s$  any bank  $j \notin K_i^*$  would be better off connecting to bank  $i$ . Hence, there is no bank  $j$  such that  $j \notin K_i^*$  if  $s_i^* = s$  and  $k_i^* < \underline{k}(\eta)$ . But then  $k_i^* = n - 1$ , a contradiction. Therefore, if  $k_i^* < \underline{k}(\eta)$  then  $s_i^* = g$  and  $k_j^* > \underline{k}(\eta)$  for any bank  $j \notin K_i^*$ .

Finally, if  $(\hat{K}, x^*, s^*)$  is not an INE it is because bank  $j$  would choose the safe project as well, once  $\hat{K}$  is given, or, when both banks  $i$  and  $j$  choose the gambling project in

$(K^*, x^*, s^*)$ , an INE would select at least one of them choosing the safe project. In all these cases, the new INE will yield an higher expected payoffs for both banks  $i$  and  $j$ . This implies that the participation constraint would be satisfied, and therefore a higher total expected payoff than in  $(\hat{K}, x^*, s^*)$ .

*Proof of Statements 2a, 2b and 2c.*

Assume that  $(K^*, x^*, s^*)$  is a CFB allocation where there is at least one bank  $i$  with  $s_i^* = g$  and  $g_i^* \neq 0$ . Let  $j \in G_i^*$ . Assume that statement 2a is not true and  $k_j > \bar{k}(\eta)$ . Take a new allocation  $(\hat{K}, x^*, s^*)$  where the bank capital and strategies are the same but the network structure severs the link between banks  $i$  and  $j$ . Formally,  $\hat{K}_i = K_i^* \setminus \{j\}$ ,  $\hat{K}_j = K_j^* \setminus \{i\}$ , and  $\hat{K}_r = K_r^*$  for all  $r \neq i, j$ .

Second, assume that statement 2b is not true and there is a bank  $r \neq j$  with  $k_r < \bar{k}(\eta)$  and  $r$  and  $j$  are not directly connected. Take a new allocation  $(\hat{K}, x^*, s^*)$  where the bank capital and strategies are the same but the network structure severs the link between banks  $i$  and  $j$  and it creates a link between banks  $j$  and  $r$ . Formally,  $\hat{K}_i = K_i^* \setminus \{j\}$ ,  $\hat{K}_j = K_j^* \setminus \{i\} \cup \{r\}$ , and  $\hat{K}_r = K_r^* \cup \{j\}$  and for  $\hat{K}_l = K_l^*$  for all  $l \neq i, j, r$ .

Finally, assume that statement 2c is not true and there is a bank  $r \neq j$  with  $k_r > \bar{k}(\eta)$  with  $s_r^* = g$  such that one of its direct gambling neighbors  $s$  is not directly connected to  $j$ . Take a new allocation  $(\hat{K}, x^*, s^*)$  where the bank capital and strategies are the same but the network structure severs the links between banks  $i$  and  $j$ , and between  $r$  and  $s$ , and it creates a link between banks  $j$  and  $s$ . Formally,  $\hat{K}_i = K_i^* \setminus \{j\}$ ,  $\hat{K}_j = K_j^* \setminus \{i\} \cup \{s\}$ ,  $\hat{K}_r = K_r^* \setminus \{s\}$  and  $\hat{K}_s = K_s^* \setminus \{r\} \cup \{j\}$  and for  $\hat{K}_l = K_l^*$  for all  $l \neq i, j, r, s$ .

We show that for all the three cases the allocation  $(\hat{K}, x^*, s^*)$  satisfies the participation constraints for any bank and it yields a higher total expected payoff. As before, if the allocation  $(\hat{K}, x^*, s^*)$  is not an INE, either bank  $i$  or bank  $j$  or both invest in the safe project in  $(\hat{K}, x^*, s^*)$ . Such allocation yields both the individual and the total expected payoffs higher than the allocation  $(K^*, x^*, s^*)$ , which then cannot be a solution to the planner's problem.

1. Participation constraints: If  $\eta < \frac{f(\bar{k}(\eta))}{f(\bar{k}(\eta) + 1)}$  then  $\eta f(k_i^*) < f(k_i^* - 1)$ , given that  $k_i^* > \bar{k}(\eta)$ , which implies that

$$\xi \eta^{g_i} f(k_i^*) R x_i^* + B < \xi \eta^{g_i - 1} f(k_i^* - 1) R x_i^* + B,$$

and

$$\xi \eta^{g_i} f(k_i^*) R < \xi \eta^{g_i - 1} f(k_i^* - 1) R.$$

Given that  $(K^*, x^*, s^*)$  satisfies the participation constraints for any bank, it has to be that  $\xi \eta^{g_i} f(k_i^*) R x_i^* + B \geq m_i^A$  and  $\xi \eta^{g_i} f(k_i^*) R \geq 1$ , and therefore  $(\hat{K}, x^*, s^*)$

satisfies the participation constraints for bank  $i$ . The proof is equivalent for bank  $j$  if statement 2a is considered and for bank  $r$  if statement 2c is considered. Notice that bank  $j$  in statements 2b and 2c and bank  $s$  in statement 2c are indifferent between the allocations  $(K^*, x^*, s^*)$  and  $(\hat{K}, x^*, s^*)$ . Consider now the participation constraint for bank  $r$  in statement 2b, we have to check it is satisfied in  $(\hat{K}, x^*, s^*)$ . Note that given the definition of  $\bar{k}(\eta)$ , it has to be that  $\eta f(k_r^*) > f(k_r^* + 1)$  for  $k_r^* < \bar{k}(\eta)$ . This implies that

$$\xi \eta^{g_r+1} f(k_r^* + 1) R x_r^* + B > \xi \eta^{g_r} f(k_r^*) R x_r^* + B,$$

and

$$\xi \eta^{g_r+1} f(k_r^* + 1) R > \xi \eta^{g_r} f(k_r^*) R,$$

if  $r$  chooses the gambling project, or

$$\eta^{g_r+1} f(k_r^* + 1) R x_r^* > \eta^{g_r} f(k_r^*) R x_r^*,$$

and

$$\eta^{g_r+1} f(k_r^* + 1) R > \eta^{g_r} f(k_r^*) R,$$

otherwise. Given that  $(K^*, x^*, s^*)$  satisfies the participation constraints for any bank, it has to be that  $\xi \eta^{g_r} f(k_r^*) R x_r^* + B \geq m_r^A$  and  $\xi \eta^{g_r} f(k_r^*) R \geq 1$  if bank  $r$  is gambling, or  $\eta^{g_r} f(k_r^*) R x_r^* \geq m_r^A$  and  $\eta^{g_r} f(k_r^*) R \geq 1$ , otherwise. Therefore the allocation  $(\hat{K}, x^*, s^*)$  satisfies the participation constraints for bank  $r$  as well.

2. Total expected payoff: Let

$$\sum_{l \in N} m_l(\hat{K}, x_l^*, s^*) - \sum_{l \in N} m_l(K^*, x_l^*, s^*) = \Delta.$$

Depending on each of the three statements, we have:

- Statement 2a.  $\Delta = \xi \eta^{g_i} R (x_i^* + 1) [f(k_i^* - 1) - \eta f(k_i^*)] + \xi \eta^{g_j} R (x_j^* + 1) [f(k_j^* - 1) - \eta f(k_j^*)]$ , where both  $k_i^* > \bar{k}(\eta)$  and  $k_j^* > \bar{k}(\eta)$ .
- Statement 2b.  $\Delta = \xi \eta^{g_i} R (x_i^* + 1) [f(k_i^* - 1) - \eta f(k_i^*)] + \xi \eta^{g_r} R (x_r^* + 1) [\eta f(k_r^* + 1) - f(k_r^*)]$ , where  $k_i^* > \bar{k}(\eta)$  and  $k_r^* < \bar{k}(\eta)$ .
- Statement 2c.  $\Delta = \xi \eta^{g_i} R (x_i^* + 1) [f(k_i^* - 1) - \eta f(k_i^*)] + \xi \eta^{g_r} R (x_r^* + 1) [f(k_r^* - 1) - \eta f(k_r^*)]$ , where both  $k_i^* > \bar{k}(\eta)$  and  $k_r^* > \bar{k}(\eta)$ .

Using the same arguments we discussed above, it follows that for the three cases we have  $\Delta > 0$ . ■

**Proof of Corollary 1.** Assume first that  $\underline{k}(\eta) = n - 1$ . By statement 1 of Proposition 5 if there is any bank  $i$  with  $k_i^* < n - 1$  then, for all bank  $j$  not directly connected to bank  $i$ , we have  $k_j^* = n - 1$ . But this is a contradiction, since any bank  $j$  with  $n - 1$  connections has to be directly connected to bank  $i$ .

Assume now that  $\bar{k}(\eta) = 0$ . By statement 2a of Proposition 5, if there is any gambling bank  $i$  directly connected to another gambling bank  $j$ , it has to be that  $k_j^* = 0$ . But again this is a contradiction with the fact that bank  $j$  is directly connected to bank  $i$ .

Finally, assume that  $\underline{k}(\eta) \neq n - 1$  and  $\bar{k}(\eta) \neq 0$ . By definition,  $\bar{k}(\eta)$  is the minimum number  $k$  such that  $\eta f(k) \leq f(k + 1)$ . Then it has to be that  $\eta f(k) > f(k + 1)$  for  $k \leq \bar{k}(\eta) - 1$ . In other words, we have that  $\eta > \frac{f(k)}{f(k+1)}$  for any  $k \leq \bar{k}(\eta) - 1$ , or  $\eta > \frac{f(k-1)}{f(k)}$  for any  $k \leq \bar{k}(\eta)$ . ■

**Proof of Proposition 6.** Assume that bank  $i$  is choosing the safe project in the CFB allocation  $(K^*, x^*, s^*)$ , with  $g_i^* > 0$ . As shown in the text, if  $\eta < \frac{1}{f(1)}$  we have  $g_j = 0$  for any  $j \in G_i$ . Take a new allocation  $(\hat{K}, \hat{x}, s^*)$  where the investment strategies are the same but (i) the bank capital given to every  $j \in G_i$  is the autarky-equivalent payoff  $x_j^A(\hat{K}, s^*)$ , and (ii) the network structure severs all the risky links of bank  $i$ . Formally, (i)  $\hat{x}_j = x_j^A(\hat{K}, s^*)$  for any  $j \in G_i$ ,  $\hat{x}_i = x_i^* + \sum_{j \in G_i} (x_j^* - \hat{x}_j)$  and  $\hat{x}_r = x_r^*$  for all  $r \notin G_i$ ,  $r \neq i$ , and (ii)  $\hat{K}_i = K_i^* \setminus \{G_i\}$ ,  $\hat{K}_j = K_j^* \setminus \{i\}$ , for all  $j \in G_i$  and  $\hat{K}_r = K_r^*$  for all  $r \notin G_i$ ,  $r \neq i$ . We show that the new allocation  $(\hat{K}, \hat{x}, s^*)$  satisfies the participation constraints for any bank, it is an INE and it yields a higher expected payoff in the economy. Therefore, the initial allocation  $(K^*, x^*, s^*)$  cannot be a solution to the planner's problem.

1. Participation constraints: We show only the case of bank  $i$ . For any bank  $j \in G_i$  the investor participation constraints are satisfied by definition. For depositors in any bank  $j \in G_i$ , recall that  $g_j^* = 0$  since  $\eta < \frac{1}{f(1)}$ . This means that depositors in any bank  $j \in G_i$  obtain  $\xi f(k_j^* - 1)R$ , which is greater than 1 given that  $\xi R \geq 1$ . For any other bank, the participation constraints are satisfied given that the allocation  $(K^*, x^*, s^*)$  satisfies the participation constraints. Depositors in bank  $i$  obtain  $f(k_i^* - g_i)R$ , which is greater than or equal to 1 given that both  $f(k_i^* - g_i)$  and  $R$  are greater than or equal to 1. For investors in bank  $i$ , we need to show that

$$x_i^* + \sum_{j \in G_i} (x_j^* - \hat{x}_j) \geq x_i^A(\hat{K}, s^*) \text{ for } \eta < \frac{1}{1 + n\rho}.$$

Recall that, given that  $(K^*, x^*, s^*)$  satisfies the participation constraint, it has to be

that  $x_i^* \geq x_i^A(K^*, s^*)$  and  $x_j^* \geq x_j^A(K^*, s^*)$  for any  $j \in G_i$ . Therefore,

$$x_i^* + \sum_{j \in G_i} (x_j^* - \hat{x}_j) \geq x_i^A(K^*, s^*) + \sum_{j \in G_i} (x_j^A(K^*, s^*) - \hat{x}_j).$$

Note that, given that  $(K^*, x^*, s^*)$  is an INE satisfying the participation constraint, by Proposition 4 any  $j \in G_i$  chooses the gamble project in autarky. This means that  $x_j^A(K^*, s^*) = \frac{e_j}{f(k_j^*)}$  and  $x_j^A(\hat{K}, s^*) = \frac{e_j}{f(k_j^* - 1)}$  for any  $j \in G_i$ . Since bank  $i$  chooses the safe project, we have  $x_i^A(K^*, s^*) = \frac{m_i^A}{\eta^{g_i} f(k_i^*) R}$  and  $x_i^A(\hat{K}, s^*) = \frac{m_i^A}{\eta^{g_i} f(k_i^* - g_i) R}$ , with  $m_i^A = \max\{R e_i, \xi R e_i + B\}$ .

Consider first  $x_j^A(K^*, s^*)$  for any  $j \in G_i$ . Since  $(K^*, x^*, s^*)$  is an optimal allocation it has to be a core-periphery structure. Furthermore,  $g_j = 0$  for any  $j \in G_i$  given that  $\eta < \frac{1}{f(1)}$ . Then,  $k_j^* \leq k_i^*$  for any  $j \in G_i$ . This is so given that (i) bank  $i$  is connected to all other banks choosing the safe project and to all banks in  $G_i$ , that is  $k_i^* = c^* - 1 + g_i$  (with  $c^*$  being the number of banks in the core); (ii) any bank  $j$  choosing the gambling project can be connected at most to all the banks choosing the safe project (and no bank choosing the gambling project), that is  $k_j^* \leq c^*$ . Then, since  $g_i \geq 1$ , we have  $k_j^* \leq k_i^*$  and therefore

$$x_j^A(K^*, s^*) = \frac{e_j}{f(k_j^*)} \geq \frac{e_j}{f(k_i^*)} \text{ for any } j \in G_i. \quad (32)$$

Consider now  $x_j^A(\hat{K}, s^*)$  for any  $j \in G_i$ . Note that

$$x_j^A(\hat{K}, s^*) = \frac{e_j}{f(k_j^* - 1)} \leq e_j \text{ for any } j \in G_i,$$

given that  $f(k_j^* - 1) \geq 1$  for any  $k_j^* \geq 1$ . Then we have

$$x_i^A(K^*, s^*) + \sum_{j \in G_i} (x_j^A(K^*, s^*) - \hat{x}_j) \geq \frac{m_i^A}{\eta^{g_i} f(k_i^*) R} + \sum_{j \in G_i} \left( \frac{e_j}{f(k_i^*)} - e_j \right). \quad (33)$$

Note that

$$\frac{m_i^A}{\eta^{g_i} f(k_i^*) R} + \sum_{j \in G_i} \left( \frac{e_j}{f(k_i^*)} - e_j \right) \geq \frac{m_i^A}{f(k_i^* - g_i) R} = x_i^A(\hat{K}, s^*),$$

or

$$m_i^A \left[ \frac{1}{\eta^{g_i} f(k_i^*) R} - \frac{1}{f(k_i^* - g_i)} \right] \geq \frac{f(k_i^*) - 1}{f(k_i^*)} \sum_{j \in G_i} e_j,$$

given that (i)  $m_i^A \left[ \frac{1}{\eta^{g_i} f(k_i^*) R} - \frac{1}{f(k_i^* - g_i)} \right] \geq \frac{B}{R} \times \frac{1 - \eta \rho}{\eta \rho}$ , by definition of  $m_i^A$  and the fact that  $1 \leq f(\cdot) \leq \rho$ , (ii)  $\frac{f(k_i^*) - 1}{f(k_i^*)} \sum_{j \in G_i} e_j \leq (n - 1) I^*(0, 0, \xi, \eta) = \frac{B}{R} \times \frac{n - 1}{1 - \eta}$ , given that by Proposition 4 we know that any  $j \in G_i$  chooses the gambling project in autarky and  $f(k_i^*) - 1 \leq f(k_i^*)$ , and (iii)  $\eta \rho < \frac{1 - \eta}{n - \eta}$  for  $\eta < \frac{1}{\rho(1 + n\rho)}$ .

2. If  $(K^*, x^*, s^*)$  is a CFB then it has to be an INE. Consider any player  $j \in G_i$ . Given that  $(K^*, x^*, s^*)$  is an INE it has to be that

$$x_j^* < \frac{B}{(1 - \xi) f(k_j^*) R},$$

for any  $j \in G_i$ , since  $s_j^* \in G_i$  and  $g_j = 0$  (recall  $\eta < \frac{1}{f(1)}$ ). From equation (32),  $x_j^* \geq \frac{e_j}{f(k_j^*)}$ . The two last inequalities imply that

$$e_j < \frac{B}{(1 - \xi) R}. \quad (34)$$

By definition of  $\hat{x}$ , we have  $\hat{x}_j = x_j^A(\hat{K}, s^*) = \frac{e_j}{f(k_j^* - 1)}$ . From (34),  $\frac{e_j}{f(k_j^* - 1)} < \frac{B}{(1 - \xi) f(k_j^* - 1) R}$ , and therefore  $\hat{x}_j < I^*(k_j^* - 1, 0, \xi, \eta)$ .

Consider now bank  $i$  and recall that  $\hat{x}_i = x_i^* + \sum_{j \in G_i} (x_j^* - \hat{x}_j)$ . From equation (33) we know that

$$x_i^* + \sum_{j \in G_i} (x_j^* - \hat{x}_j) \geq \frac{m_i^A}{\eta^{g_i} f(k_i^*) R} + \sum_{j \in G_i} \left( \frac{e_j}{f(k_i^*)} - e_j \right).$$

We prove that

$$\frac{m_i^A}{\eta^{g_i} f(k_i^*) R} + \sum_{j \in G_i} \left( \frac{e_j}{f(k_i^*)} - e_j \right) \geq I^*(0, 0, \xi, \eta) = \frac{B}{(1 - \xi) R}. \quad (35)$$

Rearranging terms, equation (35) is equivalent to

$$\frac{m_i^A}{\eta^{g_i} f(k_i^*) R} \geq \frac{B}{(1 - \xi) R} + \frac{f(k_i^*) - 1}{f(k_i^*)} \sum_{j \in G_i} e_j.$$

Note that  $m_i^A \geq B$  and  $\eta^{g_i} f(k_i^*) R \leq \eta \rho R$ . Thus,

$$\frac{m_i^A}{\eta^{g_i} f(k_i^*) R} \geq \frac{B}{\eta \rho R} \geq n \frac{B}{(1 - \eta) R},$$

the last inequality being true for  $\eta < \frac{1}{\rho(1+n\rho)}$ . Finally, we have

$$n \frac{B}{(1-\xi)R} \geq \frac{B}{(1-\xi)R} + \frac{f(k_i^*) - 1}{f(k_i^*)} \sum_{j \in G_i} e_j,$$

given that  $\frac{f(k_i^*) - 1}{f(k_i^*)} \sum_{j \in G_i} e_j < g_i \frac{B}{(n-1)R}$ , since  $e_j < I^*(0, 0, \xi, \eta)$  for any  $j \in G_i$ , and that  $g_i \leq (n-1)$ .

3. Expected total payoff: We prove that

$$\begin{aligned} \sum_{r \in N} [m_r(\hat{K}, \hat{x}, s^*) + M_r(\hat{K}, s^*)] - \sum_{r \in N} [m_r(K^*, x^*, s^*) + M_r(K^*, s^*)] &= \\ &= Rf(k_i^* - g_i)(\hat{x}_i + 1) - \eta^{g_i} f(k_i^*)R(x_i^* + 1) + \\ &+ \xi \sum_{j \in G_i} f(k_j^* - 1)R(\hat{x}_j + 1) - \xi \sum_{j \in G_i} f(k_j^*)R(x_j^* + 1) > 0, \end{aligned}$$

for  $\eta < \frac{1}{\rho(1+n\rho)}$ . Recall that  $\hat{x}_i = x_i^* + \sum_{j \in G_i} (x_j^* - \hat{x}_j)$ , where  $\hat{x}_j = \frac{e_j}{f(k_j^* - 1)}$ . Rearranging terms, we have to prove that

$$\begin{aligned} [f(k_i^* - g_i) - \eta^{g_i} f(k_i^*)](x_i^* + 1) + \sum_{j \in G_i} [f(k_i^* - g_i) - \xi f(k_j^*)] x_j^* - \\ - \sum_{j \in G_i} [f(k_i^* - g_i) - \xi f(k_j^* - 1)] \hat{x}_j > \xi \sum_{j \in G_i} [f(k_j^*) - f(k_j^* - 1)]. \end{aligned}$$

Given that  $\xi < \eta < \frac{1}{f(1)}$ , and by concavity of  $f$ ,  $\sum_{j \in G_i} [f(k_i^* - g_i) - \xi f(k_j^*)] x_j^* \geq 0$ . Furthermore,  $f(k_i^* - g_i) - \eta^{g_i} f(k_i^*) > \xi \sum_{j \in G_i} [f(k_j^*) - f(k_j^* - 1)]$  given that (i)  $k_i^* = c^* - 1 + g_i \geq c^* \geq k_j^*$ , where  $c^*$  is the number of core banks and  $g_i \geq 1$ , and (ii)  $\xi < \eta < \frac{1}{\rho(1+n\rho)}$ . So it suffices to show that

$$[f(k_i^* - g_i) - \eta^{g_i} f(k_i^*)] x_i^* > \sum_{j \in G_i} [f(k_i^* - g_i) - \xi f(k_j^* - 1)] \hat{x}_j$$

to prove our claim.

We have (i)  $[f(k_i^* - g_i)] - \eta^{g_i} f(k_i^*) \geq (1 - \eta\rho)$ ; (ii)  $x_i^* \geq \frac{B}{(1-\xi)\eta\rho R}$ , since  $x^*$  satisfies the incentive constraint; (iii)  $f(k_i^* - g_i) - \xi f(k_j^* - 1) < \rho$  for any  $j \in G_i$ ; (iv)  $\hat{x}_j = \frac{e_j}{f(k_j^* - 1)} \leq e_j < \frac{B}{(1-\xi)R}$ , given that by Proposition 4, bank  $j$  chooses the gambling project in autarky. Then the following holds

$$[f(k_i^* - g_i) - \eta^{g_i} f(k_i^*)] x_i^* \geq (1 - \eta\rho) \frac{B}{(1-\xi)\eta\rho R} > (n-1)\rho \frac{B}{(1-\xi)R} > \sum_{j \in G_i} [f(k_i^* - g_i) - \xi f(k_j^* - 1)]$$

where the second inequality holds under the assumption  $\eta < \frac{1}{\rho(1+n\rho)}$ . ■

**Proof of Proposition 7.** Assume by contradiction that  $(K^e, e, s^e)$  is a DEWT, but there are two banks  $i$  and  $j$  such that  $s_i^e = s_j^e = b$  but  $i \notin K_j^e$ , and therefore  $j \notin K_i^e$ . We prove that  $(K^e \cup ij, e, s^e)$  is an INE and that both  $m_i(K^e \cup ij, e, s^e) > m_i(K^e, e, s^e)$  and  $m_j(K^e \cup ij, e, s^e) > m_j(K^e, e, s^e)$ , contradicting the fact that  $(K^e, e, s^e)$  is a PSWT, and therefore it cannot be a DEWT.

Note that, since  $(K^e, e, s^e)$  is a DEWT, it has to be an INE. This means that  $e_i \geq I^*(k_i, g_i, \xi, \eta)$  and  $e_j \geq I^*(k_j, g_j, \xi, \eta)$ . Furthermore, it has to be  $I^*(k_i, g_i, \xi, \eta) \geq I^*(k_i + 1, g_i, \xi, \eta)$  and  $I^*(k_j, g_j, \xi, \eta) \geq I^*(k_j + 1, g_j, \xi, \eta)$  given that  $f(k)$  is increasing in  $k$ . This implies that  $e_i \geq I^*(k_i + 1, g_i, \xi, \eta)$  and  $e_j \geq I^*(k_j + 1, g_j, \xi, \eta)$ . Therefore,  $(K^e \cup ij, e, s^e)$  is also an INE. Finally, note that

$$m_i(K^e \cup ij, e, s^e) = \eta^{g_i} f(k_i + 1) Re_i > \eta^{g_i} f(k_i) Re_i = m_i(K^e, e, s^e),$$

and

$$m_j(K^e \cup ij, e, s^e) = \eta^{g_j} f(k_j + 1) Re_j > \eta^{g_j} f(k_j) Re_j = m_j(K^e, e, s^e),$$

since the function  $f(k)$  is increasing in  $k$ . Therefore,  $(K^e, e, s^e)$  cannot be a DEWT. ■

**Proof of Proposition 8.** Assume that  $k_i^e < \underline{k}(\eta)$ . This means that  $k_i^e \leq \underline{k}(\eta) - 1$ , and by definition of  $\underline{k}(\eta)$  and the fact that the ratio  $\frac{f(k)}{f(k+1)}$  is increasing in  $k$ , we have  $\eta \geq \frac{f(k_i^e)}{f(k_i^e+1)}$ . Hence, bank  $i$  will gain if connecting to any other bank  $j \notin K_i^e$ , no matter  $j$ 's strategy. Therefore,  $K^e$  can be a pairwise stable structure only if there is no other bank  $j$  willing to connect to bank  $i$ . If bank  $i$  is choosing the safe project, any bank not yet connected to bank  $i$  would be better off by connecting to it. Therefore, if  $s_i = b$  then  $K_i^e = N \setminus \{i\}$ . Otherwise, if  $s_i^e = g$  it has to be that no other bank  $j$  not connected to bank  $i$  could be better off from connecting to bank  $i$ . This could only happen if  $k_j^e \geq \underline{k}(\eta)$  for any bank  $j \notin K_i^e$ , given that any bank  $j$  would like to connect with bank  $i$  if  $k_j^e < \underline{k}(\eta)$ .

Assume that  $k_i^e > \bar{k}(\eta)$  for some bank  $i$ . This means that  $k_i^e \geq \bar{k}(\eta) + 1$ , and by definition of  $\bar{k}(\eta)$  and the fact that the ratio  $\frac{f(k)}{f(k+1)}$  is increasing in  $k$ , we know that  $\eta < \frac{f(k_i^e-1)}{f(k_i^e)}$ . This means that bank  $i$  is better off if it unilaterally disconnects any of its risky links. So the allocation  $(K^e, e, s^e)$  can only be PSWT if  $G_i = \emptyset$ . ■

**Proof of Corollary 2.** The proof is similar to the one of Corollary 1 and therefore omitted. ■

**Proof of Proposition 9.** Given the definition of  $Q$  we have

$$Q = \{i \text{ such that } I^*(k_i, \widehat{g}_i, \xi, \eta) \leq x_i < I^*(k_i, \widehat{g}_i + q_i, \xi, \eta)\}$$

where  $\widehat{g}_i$  is the number of banks in  $K_i \setminus Q$  that choose the gambling project, and  $q_i$  is the minimum number of banks in  $Q$  connected to bank  $i$  that, by choosing the gambling project, would make bank  $i$  switch from the safe project to the gambling project. Note that  $1 \leq q_i \leq |K_i \cap Q|$ . Namely,  $q_i$  has to be at least one, otherwise bank  $i$  will always choose the gambling project and would not belong to  $Q(s, s')$ . Furthermore,  $q_i$  has to be at most  $|Q \cap K_i|$ , i.e., the number of banks in  $Q$  that are connected to bank  $i$ . Otherwise, bank  $i$  will always choose the safe project and would not belong to  $Q(s, s')$ . Assume that the sequential-move investment game calls bank  $i$  to make the investment decision (given the choices of banks  $Q$ .) If history in the game is such that  $q_i$  banks in  $Q$  connected to bank  $i$  choose the gambling project, bank  $i$  chooses the gambling project as well. If history in the game is such that there are less than  $q_i$  banks in  $Q$  connected to bank  $i$  choosing the gambling project, bank  $i$  chooses the safe project. But if history is such that if by choosing the safe project there are less than  $q_i$ , say,  $c_1$  banks choosing the gambling project in  $Q$  and if by choosing the gambling project there are more than  $q_i$ , say,  $c_2$  banks choosing the gambling project, bank  $i$  chooses the gambling asset only if

$$\eta^{\widehat{g}_i + c_1} f(k_i) R x_i < \xi \eta^{\widehat{g}_i + c_2} f(k_i) R x_i + B, \quad (36)$$

where  $c_1 < q_i \leq c_2$ . The previous inequality implies that

$$x_i < \frac{B}{(1 - \xi \eta^{c_2 - c_1}) \eta^{\widehat{g}_i + c_1} f(k_i) R}, \quad (37)$$

a contradiction with the fact that

$$x_i \geq \frac{B}{(1 - \xi) \eta^{\widehat{g}_i + c_1} f(k_i) R},$$

given that  $c_1$  banks in  $|Q \cap K_i|$  are not enough to make bank  $i$  switching from the safe project to the gambling one.

This implies that every time a bank is critical in the rule of order to decide among different continuation paths, it will choose the safe path. Then we can conclude that for any rule of order, the sequential-move investment game selects the INE profiles where the highest number of banks choose the safe project. ■

**Proof of Proposition 10.** Recall that  $x_i = e_i + t_i \geq 0$ . Assume bank  $i$  is considering to make a transfer  $t_i$ . Since  $t_i \geq 0$  it has to be that  $t_i \leq x_i$ . Denote by  $j$  the bank that receives a transfer  $t_j$  to be induced to choose the safe project. Therefore  $x_j + t_j \geq I^*(k_j, g_j, \xi, \eta)$ . Given that  $t_i \geq t_j$  it has to be that  $x_i + x_j \geq I^*(k_j, g_j, \xi, \eta) > \frac{B}{(1 - \xi)\rho R}$ . Clearly,  $x_i + x_j \leq E = \sum_{i \in N} e_i$ . Thus, for  $E < \frac{B}{(1 - \xi)\rho R}$  there is no transfer  $t_i$  that can convince bank  $j$  to invest in the safe project, no matter the values of  $x_i$  or  $x_j$ . This happens when  $\xi > 1 - \frac{B}{\rho R E}$ . We have  $\bar{\xi} = \max\{1 - \frac{B}{\rho R E}, 0\}$ . ■

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